Boundedness in Finite Dimensional n-Normed Spaces

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Abstract: Sukaesih and Gunawan, in 2016, shown relation of bounded set in n-normed spaces through the equivalence of norm and n-norm. In this paper, the relation of boundedness with respect to m linearly independent vectors ($m \geq n$) are shown by the relation of linearly independent sets.

1 INTRODUCTION

Gähler was introduced 2-normed spaces, in 1964. The generalization to n-normed spaces also done by Gähler (Gähler , 1969). An n-norm is a real function $\|\cdot, \dots, \cdot\|: X^n \rightarrow [0, \infty)$ which satisfies the following conditions for all $x, x_1, \dots, x_n \in X$ and for any $\alpha \in \mathbb{R}$,

- $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ linearly dependent,
- $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,
- $\|\alpha\, x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$ for every $\alpha \in \mathbb{R}$,
- $\|x_1 + x, x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x, x_2, \dots, x_n\|$.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

In 2011, Harikrishnan and Ravindra introduced the definition of bounded set in 2-normed spaces. It was than generalized to definition of bounded set in n-normed spaces by Kir and Kiziltunc (Kir and Kiziltunc, 2014). But, Gunawan *et.al.* (Gunawan *et.al.*, 2016) found lack of the Kir and Kiziltunc's definition, they then defined new definition of bounded set in n-normed spaces.

Definition 1: (Sukaesih, 2017) Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space, B be a nonempty subset of X and $\mathcal{A} = \{a_1, \cdots, a_m\}$ be a linearly independent set ($m \geq n$). Then B is called bounded with respect to $\mathcal{A}$ *if there is* $M > 0$ *such that*

$$\left\| x, a_{i_2}, \cdots, a_{i_n} \right\| \leq M$$

for every $x \in B$ and for every $\{i_2, \cdots, i_n\} \subset \{1, \cdots, m\}$.

Let $\mathfrak{B}_{\mathcal{A}}(X, \|\cdot, \dots, \cdot\|)$ be a collection of bounded set with respect to $\mathcal{A}$. If a set B is bounded with respect to $\mathcal{A}$, then $B \in \mathfrak{B}_{\mathcal{A}}(X, \|\cdot, \dots, \cdot\|)$.

Hereafter, let $(X, \|\cdot, \cdots, \cdot\|)$ be a finite dimensional n-normed space (dim(X)= d), B be a nonempty set of X, and $\mathcal{A} = \{a_1, \cdots, a_m\}$ be a linearly independent vectors in X (rank($\mathcal{A}$) $= m$), where $n \leq m \leq d$.

Sukaesih and Gunawan (Sukaesih and Gunawan, 2016) shown the relation of boundedness with respect to any linearly independent sets.

Lemma 2: (Sukaesih and Gunawan, 2016) Let (X, $\|\cdot, \cdots, \cdot\|$) be a finite dimensional n-normed space, B be a nonempty set of X. If $\mathcal{A}_1 = \{a_{11}, \cdots, a_{1m_1}\}$ be a linearly independent set in X ($m_1 = n$ or $m_1 = d$) and $\mathcal{A}_2 = \{a_{21}, \cdots, a_{2m_2}\}$ be a linearly independent set in X ($m_2 = n$ or $m_2 = d, \mathcal{A}_1 \neq \mathcal{A}_2$) then a set B is bounded with respect to $\mathcal{A}_1$ if and only if B is bounded with respect to $\mathcal{A}_2$.

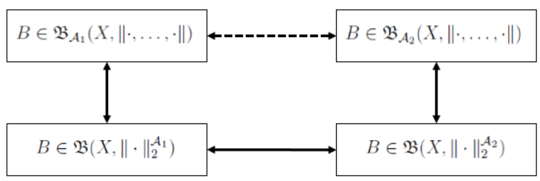

Figure 1: The relation between the boundedness with respect to $\mathcal{A}_1$ and the boundedness with respect to $\mathcal{A}_2$.

Lemma 2 was proven by following Corollary and equivalencies in normed spaces.

Corollary 3: (Sukaesih and Gunawan, 2016) Let $(X, \|\cdot, \cdots, \cdot\|)$ be a finite dimensional n-normed

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space (dim(X)= d) which also equipped with a norm $\|\cdot\|_{\mathcal{A}}$, B be a nonempty set of X. If $\mathcal{A} = \{a_1, \cdots, a_n\}$ be a linearly independent set in X then a set B is bounded with respect to $\mathcal{A}$ if and only if B is bounded in (X, $\|\cdot\|_{\mathcal{A}}$).

Let $\mathfrak{B}(X, \|\cdot\|_{\mathcal{A}})$ be collection of bounded set in $(X, \|\cdot\|_{\mathcal{A}})$. If a set B is bounded in $(X, \|\cdot\|_{\mathcal{A}})$, then $B \in \mathfrak{B}(X, \|\cdot\|_{\mathcal{A}})$. Further on norm $\|\cdot\|_{\mathcal{A}}$ could be studied in (Burhan, 2011).

2 MAIN RESULT

In an n-normed space $(X, \|\cdot, \cdots, \cdot\|)$, we have many sets of m linearly independent vectors in X ($n \leq m \leq d$). Here some relation of boundedness in finite dimensional n-normed spaces.

Theorem 4: Let $(X, \|\cdot, \cdots, \cdot\|)$ be a finite dimensional n-normed space (dim(X)= d) and $\mathcal{A}_1 = \{a_1, \cdots, a_{m_1}\}$, $\mathcal{A}_2 = \{b_1, \cdots, b_{m_2}\}$ be linearly independent sets in X that span($\mathcal{A}_1$) $\subset$ span($\mathcal{A}_2$) and $n \leq m_1 \leq m_2 \leq d$. If a set B is bounded with respect to $\mathcal{A}_2$, then B is bounded set with respect to $\mathcal{A}_1$.

Proof: Because of the boundedness with respect to $\mathcal{A}_2$ then we have $\|x, b_{l_2}, \dots, b_{l_n}\| \leq M$ for every $x \in B$ and for every $\{l_2, \dots, l_n\} \subset \{1, \dots, m_2\}$. Because of $span\ (\mathcal{A}_1) \subset span\ (\mathcal{A}_2)$ then we have $a_i = \sum_{l=1}^{m_2} \alpha_{il} b_l$, such that

$$\begin{aligned}
&\|x, a_{i_2}, \dots, a_{i_n}\| \\
&= \left\| x, \sum_{l_2=1}^{m_2} \alpha_{i_2 l_2} b_{l_2}, \sum_{l_3=1}^{m_2} \alpha_{i_3 l_3} b_{l_3}, \dots, \sum_{l_n=1}^{m_2} \alpha_{i_n l_n} b_{l_n} \right\| \\
&\leq \sum_{l_2=1}^{m_2} \alpha_{i_2 l_2} \left\| x, b_{l_2}, \sum_{l_3=1}^{m_2} \alpha_{i_3 l_3} b_{l_3}, \dots, \sum_{l_n=1}^{m_2} \alpha_{i_n l_n} b_{l_n} \right\| \\
&\leq \sum_{l_2=1}^{m_2} \alpha_{i_2 l_2} \left[\sum_{l_3=1}^{m_2} \alpha_{i_3 l_3} \left\| x, b_{l_2}, b_{l_3}, \dots, \sum_{l_n=1}^{m_2} \alpha_{i_n l_n} b_{l_n} \right\| \right] \\
&\leq \sum_{l_2=1}^{m_2} \alpha_{i_2 l_2} \\
&\left[\sum_{l_3=1}^{m_2} \alpha_{i_3 l_3} \left[\dots \left[\sum_{l_n=1}^{m_2} \alpha_{i_n l_n} \|x, b_{l_2}, b_{l_3}, \dots, b_{l_n}\| \right] \dots \right] \right] \\
&\leq SM,
\end{aligned}$$

with $S = (m_2)^n \max\{\alpha_{i_2 l_2}, \dots, \alpha_{i_n l_n}\}$ for $l_2 = 1, \dots, m_2$, ..., $l_n = 1, \dots, m_2$ and for every $\{i_2, \dots, i_n\} \subset \{1, \dots, m_1\}$. □

A vector space can be generated by many linearly independent sets. If two linearly independent sets generated the same space, then the boundedness with respect to a linearly independent set tie up the boundedness with respect to another linearly independent set. Then Lemma 2 was generalized for any m with $n \leq m \leq d$.

Lemma 5: Let $(X, \|\cdot, \cdots, \cdot\|)$ be a finite dimensional n-normed space (dim(X)= d) and $\mathcal{A}_1 = \{a_1, \cdots, a_m\}$, $\mathcal{A}_2 = \{b_1, \cdots, b_m\}$ be linearly independent sets in X such that span($\mathcal{A}_1$) = span($\mathcal{A}_2$) and $n \leq m \leq d$. A set B is bounded with repect to $\mathcal{A}_2$ if and only if a set B is bounded with respect to $\mathcal{A}_1$.

Proof. From the boundedness with respect to $\mathcal{A}_2$, we have $\|x, b_{l_2}, \dots, b_{l_n}\| \leq M$ for every $x \in B$ and for every $\{l_2, \dots, l_n\} \subset \{1, \dots, m\}$. Since $span\ (\mathcal{A}_1) = span\ (\mathcal{A}_2)$ then we have $a_i = \sum_{l=1}^{m} \alpha_{il} b_l$, such that

$$\begin{aligned}
&\|x, a_{i_2}, \dots, a_{i_n}\| \\
&= \left\| x, \sum_{l_2=1}^{m} \alpha_{i_2 l_2} b_{l_2}, \sum_{l_3=1}^{m} \alpha_{i_3 l_3} b_{l_3}, \dots, \sum_{l_n=1}^{m} \alpha_{i_n l_n} b_{l_n} \right\| \\
&\leq \sum_{l_2=1}^{m} \alpha_{i_2 l_2} \left\| x, b_{l_2}, \sum_{l_3=1}^{m} \alpha_{i_3 l_3} b_{l_3}, \dots, \sum_{l_n=1}^{m} \alpha_{i_n l_n} b_{l_n} \right\| \\
&\leq \sum_{l_2=1}^{m} \alpha_{i_2 l_2} \left[\sum_{l_3=1}^{m} \alpha_{i_3 l_3} \left\| x, b_{l_2}, b_{l_3}, \dots, \sum_{l_n=1}^{m} \alpha_{i_n l_n} b_{l_n} \right\| \right] \\
&\leq \sum_{l_2=1}^{m} \alpha_{i_2 l_2} \\
&\left[\sum_{l_3=1}^{m} \alpha_{i_3 l_3} \left[\dots \left[\sum_{l_n=1}^{m} \alpha_{i_n l_n} \|x, b_{l_2}, b_{l_3}, \dots, b_{l_n}\| \right] \dots \right] \right] \\
&\leq SM,
\end{aligned}$$

with $S = (m)^n \max\{\alpha_{i_2 l_2}, \dots, \alpha_{i_n l_n}\}$ for $l_2 = 1, \dots, m$, ..., $l_n = 1, \dots, m$ and for every $\{i_2, \dots, i_n\} \subset \{1, \dots, m\}$.

Conversely, use the same way.

$$\boxed{B \in \mathfrak{B}_{\mathcal{A}_1}(X, \|\cdot, \dots, \cdot\|)} \longleftrightarrow \boxed{B \in \mathfrak{B}_{\mathcal{A}_2}(X, \|\cdot, \dots, \cdot\|)}$$

Figure 2: The relation between any two linearly independent set that generated the same space.

For $\mathcal{A}_1$ and $\mathcal{A}_2$ are basis of X, we have the following condition.

Corollary 6: Let $(X, \|\cdot,\cdots,\cdot\|)$ be a finite dimensional n-normed space ($\dim(X)= d$) and $\mathcal{B}_1, \mathcal{B}_2$ be basis on X. A set B is bounded with repect to $\mathcal{B}_2$ if and only if set B is bounded with respect to $\mathcal{B}_1$.

Proof. Use Lemma 5 for $\mathcal{A}_1$ and $\mathcal{A}_2$ are basis of X.□

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