Graphs with Partition Dimension 3 and Locating-chromatic Number 4

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Abstract: The characterization study of all graphs with partition dimension either $2, n-2, n-1$ or n has been completely done. In the case of locating-chromatic numbers, the efforts in characterizing all graphs with locating-chromatic number either $2, 3, n-1$ or n have reached to complete results. In this paper we present methods to obtain a family of graphs having partition dimension 3 or locating-chromatic number 4 by using the previous known results.

1 INTRODUCTION

The concepts of partition dimension and locating-chromatic number of connected graphs were introduced by Chartrand et al. in (Chartrand et al., 1998) and in (Chartrand et al., 2002), respectively. The locating-chromatic number for graphs is a special case of the partition dimension notion. In order to generalize these two concepts, Haryeni et al. in (Haryeni et al., 2017) enlarged the notion of the partition dimension so that it can be applied also to disconnected graphs, and Welyyanti et al. in (Welyyanti et al., 2014) enlarged the notion of locating-chromatic number for disconnected graphs.

Let $F = (V, E)$ be a (not necessarily connected) graph and $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of $V(F)$, where S_i is a *partition class* of Π for each $1 \leq i \leq k$. If the distance $d_F(v, S_i) < \infty$ for all $v \in V(F)$ and $S_i \in \Pi$, then the *representation* $r(v|\Pi)$ of v with respect to Π is $(d_F(v, S_1), d_F(v, S_2), \ldots, d_F(v, S_k))$. The partition Π is a *resolving partition* of F if every two distinct vertices $u, v \in V(F)$ have distinct representations with respect to Π, namely $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of F, denoted by $pd(F)$ for a connected F or by $pdd(F)$ for a disconnected F, is the cardinality of a smallest resolving partition of F. For a disconnected graph F, if there is no a resolving partition of F, then $pdd(F) = \infty$. In addition, if the partition Π is induced by a proper k −coloring c, then we define the color code $c_\Pi(v)$ of a vertex $v \in V(F)$ with $(d_F(v, S_1), d_F(v, S_2), \ldots, d_F(v, S_k))$. If all vertices of F have different color codes, then c is a *locating coloring* of F.

The *locating-chromatic number* of F, denoted by $\chi_L(F)$ for a connected F or by $\chi_L{}'(F)$ for a disconnected graph F, is the least integer k such that F admits a locating k −coloring. Otherwise, we say that $\chi'_L(F) = \infty$.

Chartrand et al. in (Chartrand et al., 2000) characterized all connected graphs on $n(\geq 3)$ vertices having the partition dimension $2, n$, or $n-1$. Tomescu in (Tomescu, 2008) showed that there are only 23 connected graphs on $n(\geq 9)$ vertices with the partition dimension $n-2$. Further results of the partition dimension of graphs for some graph operations, namely corona product, Cartesian product and strong product, can be observed in (Rodr´ıguez-Velazquez et al., 2016; Yero et al., 2014; Yero et al., 2010).

On the other hand, all connected graphs on n vertices with locating-chromatic number n or $n-1$ was characterized in (Chartrand et al., 2003). In the same paper, they also gave conditions for graph F on $n(\geq 5)$ vertices with $\chi_L(F) \leq n-2$. The characterization of all graphs with locating-chromatic number 3 can be seen in (Baskoro and Asmiati, 2013) and (Asmiati and Baskoro, 2012).

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In this paper, motivated by the results of the characterization of all graphs with locating-chromatic number 3, we present some method to extend those graphs so that their partition dimension is equal to 3. Furthermore, we show that these new graphs have locating-chromatic number 4. We also construct some classes of graphs by connecting some vertices in a disjoint union of paths, so that the partition dimension of these graphs remains equal to 3.

In order to present the results, we need additional notions and some known results as follows. Let $\Pi = \{S_1, S_2, \dots, S_k\}$ be a resolving partition of $V(F)$. For an integer $k \geq 1$, a vertex $u \in V(F)$ is defined as k −distance vertex with respect to Π if $d_F(v, S_i) = 0$ or k for any $S_i \in \Pi$. Note that in the locating-chromatic number of F, the only possible value of k is 1 and the vertex u satisfies this condition is called a *dominant vertex*.

Definition 1.1. *(Haryeni et al., 2019) Let F be a graph and $\Pi = \{S_1, S_2, \dots, S_k\}$ be a minimum resolving partition of F. Two distinct vertices $p, q \in V(F)$ in S_i for some $i \in [1, k]$ are called independent vertices with respect to Π if there exist two distinct integers, namely j and l which different from i , such that $d_F(p, S_j) - d_F(q, S_j) \neq d_F(p, S_l) - d_F(q, S_l)$. Furthermore, if there exists a minimum resolving partition of F such that any two vertices in the same class partition are independent, then F is called an independent graph. Otherwise, F is a dependent graph.*

Definition 1.2. *(Haryeni et al., 2019) Let F be a graph and $B \subseteq V(F)$ where $B = (b_1, b_2, \dots, b_k)$. We denote $F[(b_1, b_2, \dots, b_k); (n_1, n_2, \dots, n_k)]$ as a hair graph of F with respect to B which is obtained from F by attaching a path P_{n_i} with $n_i (\geq 2)$ vertices to a root vertex b_i, for all $i \in [1, k]$. Furthermore, the set of all hair graphs obtained from the graph F is denoted by $Hair(F)$.*

Theorem 1.3. *(Haryeni et al., 2019) Let F be a graph with a finite partition dimension. For any $H \in Hair(F)$, then*

$$pdd(H) \leq \begin{cases} pdd(F), & if\ F\ is\ independent, \\ pdd(F) + 1, & if\ F\ is\ dependent. \end{cases}$$

Proposition 1.4. *(Haryeni et al., 2019) For any $n \geq 3$, a path P_n is a dependent graph with any resolving 2-partition.*

Theorem 1.5. *(Haryeni et al., 2017) Let F be a disjoint union of m paths with different lengths. If $m = 1$, then $pd(F) = 2$. Otherwise, $pdd(F) = 3$.*

Corrollary 1.6. *(Haryeni et al., 2017) If $H \in Hair(P_m)$ and $H \ncong P_n$ for any $n \geq m$, then $pd(H) = 3$.*

2 TREES WITH PARTITION DIMENSION 3 AND LOCATING-CHROMATIC NUMBER 4

Let T be a tree with 3 dominant vertices x, y and z , and $aP_b = (a, x, u_1, u_2, u_{r-1}, u_r = y, v_1, v_2, \dots, v_{s-1}, v_s = z, b)$ be a path with r, s odd. If $r, s > 1$, then define $u^* = u_{\lfloor \frac{r}{2} \rfloor}$, $u^{**} = u_{\lfloor \frac{r+1}{2} \rfloor}$, $v^* = v_{\lfloor \frac{s}{2} \rfloor}$, and $v^{**} = v_{\lfloor \frac{s+1}{2} \rfloor}$. Note that all internal vertices u_i and v_j of T excluding u^*, u^{**}, v^* and v^{**} have degree 2. In The following result, Baskoro and Asmiati (Baskoro and Asmiati, 2013) characterized all trees with locating-chromatic number 3.

Lemma 2.1. *(Baskoro and Asmiati, 2013) In any tree T with $\chi_L(T) = 3$, the color code of any vertex is (a_1, a_2, a_3) where $\{a_1, a_2, a_3\} = \{0, 1, k\}$ for some $k \geq 1$.*

Theorem 2.2. (Baskoro and Asmiati, 2013) Let T be a tree on $n (\geq 3)$ vertices. The value $\chi_L(T) = 3$ iff T is isomorphic to either $P_3, P_4, S_{1,2}, S_{2,2}$, or any subtree in Figure 1 containing a path aP_b.

Theorem 2.3. *Let F be a graph other than a path with $\chi_L{}'(T) = 3$. Then, F is always independent.*

Proof. Let $c: V(F) \to \{1, 2, 3\}$ be any 3-coloring on graph F. Let $\Pi = \{S_1, S_2, S_3\}$ be the partition of F induced by c. We will show that any two vertices $p, q \in S_a$ for some $a \in [1, 3]$ are independent vertices with respect to Π. If p and q are in different partition classes, then certainly p and q are independent. Now assume that p and q are in the same class, say $p, q \in S_1$. This implies that $c_\Pi(x) = (0, c_2, c_3)$ and $c_\Pi(y) = (0, d_2, d_3)$. By Lemma 2.1, then we have $c_\Pi(x) = (0, c_2, 1)$ and $c_\Pi(y) = (0, d_2, 1)$, or $c_\Pi(x) = (0, 1, c_3)$ and $c_\Pi(y) = (0, 1, d_3)$, or $c_\Pi(x) = (0, 1, c_3)$ and $c_\Pi(y) = (0, d_2, 1)$, or $c_\Pi(x) = (0, c_2, 1)$ and

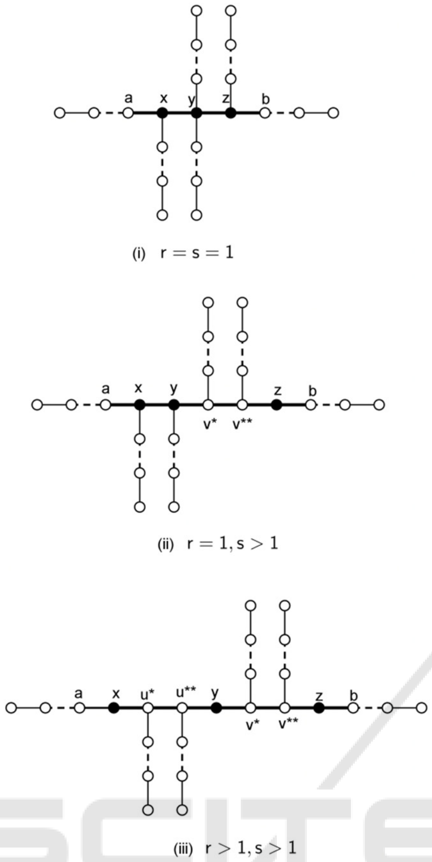

Figure 1: Any subtree T containing a path aP_b with $\chi_L(T) = 3$ Now, we present the following results.

$c_\Pi(y) = (0,1,d_3)$. Note that c is a locating coloring of F so that $c_\Pi(x) \neq c_\Pi(y)$. Therefore, for the previous four cases we can conclude that $c_2 - d_2 \neq c_3 - d_3$. This implies that any two vertices $x, y \in S_a$ of F are indepedent vertices so that F is an independent graph with respect to the partition Π.

By Theorems 2.2 and 2.3, we show that any hair graph of tree T in Theorem 2.2 has partition dimension 3 and locating-chromatic number 4.

Corollary 2.4. *Let T be either a path P_3 or P_4, a double star $S_{1,2}$ or $S_{2,2}$, or a subtree in Figure 1 containing a path aP_b. For all $H \in Hair(T)$ where $H \ncong P_m$, then $pd(H) = 3$ and with $\chi_L(H) = 4$.*

Proof. If T is isomorphic to a path, then $pd(H) = 3$ for any $H \in Hair(T)$ with $H \ncong P_m$, by Corollary 1.5. Now we suppose that T is not isomorphic to a path. Since $H \ncong P_m$, $pd(H) \geq 3$. By Theorems 2.2 and 2.3, then T is an independent graph with locating 3-coloring. By Theorem 1.3, then $pd(H) \leq pd(T) \leq \chi_L(T) = 3$. Further-more, since all trees T with $\chi_L(T) = 3$ are only a path P_3 or P_4, a double star $S_{1,2}$or $S_{2,2}$, or a subtree in Figure 1 containing a path aP_b, $\chi_L(H) \geq 4$. The coloring of H with 4 colors is given in Figure 2. The color of the new vertices of H are 4 and i alternately, where i is the color of the root vertex.

3 GRAPHS CONTAINING CYCLE WITH PARTITION DIMENSION 3 AND LOCATING-CHROMATIC NUMBER 4

In the following theorem, all graphs containing cycle with locating-chromating number 3 have been characterized, see (Asmiati and Baskoro, 2012).

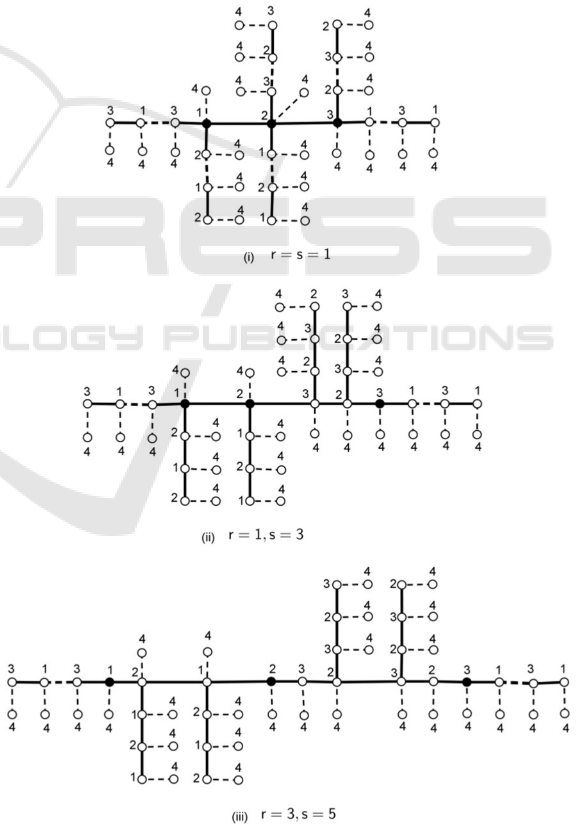

Figure 2: The locating 4-coloring of graph $H \in Hair(T)$ where T depicted in Figure 1

Theorem 3.1. (Asmiati and Baskoro, 2012) Let F be any graph having a smallest odd cycle C. Then $\chi_L(F) = 3$ iff F is a subgraph of one of the graphs in Figure 3 which every vertex $a \notin C$ of degree 3 must be lie in a path connecting two different vertices in C.

By a similar reason to Corollary 2.4, we show that for every $H \in Hair(F)$, where F is a graph in Theorem 3.1, then $pd(H) = 3$ and $\chi_L(H) = 4$.

Corollary 3.2. Let F be any graph having a smallest odd cycle C, where F is a subgraph of one of the graphs in Figure 3 which every vertex a $a \notin C$ of degree 3 must be lie in a path connecting two different vertices in C. For all $H \in Hair(F)$, then $pd(H) = 3$ and $\chi_L(H) = 4$.

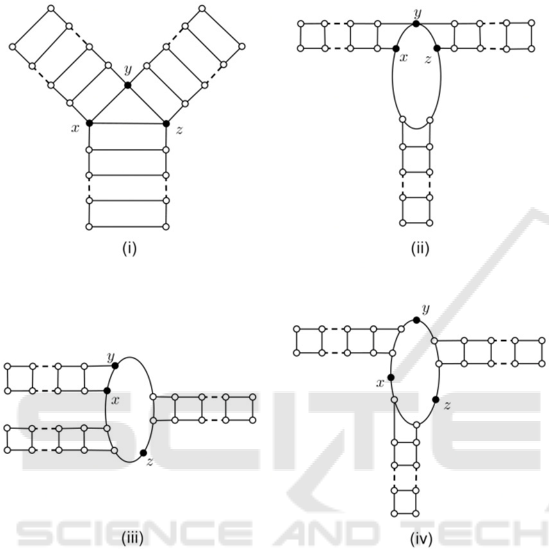

Figure 3: The four types of maximal graphs containing an odd cycle with chromatic location number 3.

For now on, for any integer $m \geq 2$, define the graph $G = \bigcup_{i=1}^{m} P_{n_i}$ where $n_1 \geq 3$ and $n_{i+1} = n_i + 1$ for all $i \in [1, m-1]$. Note that $pdd(G) = 3$ by Lemma 1.5. In the next result, we construct some graphs obtaining from disjoint union of paths $G = \bigcup_{i=1}^{m} P_{n_i}$ so that their partition dimensions remains equal to 3. Let the set of vertices and edges of G by

$V(G) = \{v_{i,j}: 1 \leq i \leq m, 1 \leq j \leq n_i\}$ and
$E(G) = \{v_{i,j}v_{i,j+1}: 1 \leq i \leq m, 1 \leq j \leq n_i - 1\}$,

respectively. Let S_1, S_2 and S_3 be three subsets of $V(G)$ where

$$S_1 = \{v_{i,1}: 1 \leq i \leq m\}, \quad (1)$$
$$S_2 = \{v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i+1\}, \quad (2)$$
$$S_3 = \{v_{i,j}: 1 \leq i \leq m, i+2 \leq j \leq n_i\}. \quad (3)$$

By the above definitions, for three distinct vertices $x, y, z \in V(F)$ where $x = v_{i,1} \in S_1$, $y = v_{i,j} \in S_2$ and $z = v_{i,k} \in S_3$ for some $i \in [1, m], j \in [2, i+1]$ and $k \in [i+2, n_i]$, we have

$$d_G(x, S_t) = \begin{cases} 0, & \text{if } t = 1, \\ 1, & \text{if } t = 2, \\ i+1, & \text{if } t = 3, \end{cases}$$
$$d_G(y, S_t) = \begin{cases} j-1, & \text{if } t = 1, \\ 0, & \text{if } t = 2, \\ i+2-j, & \text{if } t = 3, \end{cases}$$
$$d_G(z, S_t) = \begin{cases} k-1, & \text{if } t = 1, \\ k-i-1, & \text{if } t = 2, \\ 0, & \text{if } t = 3. \end{cases}$$

Now, define new graphs $G' = G \cup E_1 \cup E_2$ and $G \subseteq G'' \subseteq G'$, where E_1 and E_2 are two sets of additional edges connecting some vertices of F as follows.

$$E_1 = \{v_{i,j}v_{i+1,j}: 1 \leq i \leq m-1, 1 \leq j \leq n_i\}$$
$$E_2 = \{v_{i,j}v_{i+1,j+1}: 1 \leq i \leq m-1, 1 \leq j \leq n_i - 1\}$$

By the above definitions, then $V(G'') = V(G') = V(G)$. Let S_1, S_2 and S_3 be three subsets of $V(G'')$ similar to the equations in (1), (2) and (3), respectively. Therefore, for three distinct vertices $x, y, z \in V(G'')$ where $x = v_{i,1} \in S_1$, $y = v_{i,j} \in S_2$ and $z = v_{i,k} \in S_3$ where $i \in [1, m]$, $j \in [2, i+1]$ and $k \in [i+2, n_i]$, we have

$$\begin{aligned} d_{G''}(x, S_3) &= \min\{d_{G''}(v_{i,1}, v_{l,l+2}): 1 \leq l \leq m\} \\ &= d_{G''}(v_{i,1}, v_{i,i+2}) \\ &= i+1, \\ d_{G''}(y, S_1) &= \min\{d_{G''}(v_{i,j}, v_{l,1}): 1 \leq l \leq m\} \\ &= d_{G''}(v_{i,j}, v_{i,1}) \\ &= j-1, \\ d_{G''}(y, S_3) &= \min\{d_{G''}(v_{i,j}, v_{l,l+2}): 1 \leq l \leq m\} \\ &= d_{G''}(v_{i,j}, v_{i,i+2}) \\ &= i+2-j, \\ d_{G''}(z, S_1) &= \min\{d_{G''}(v_{i,k}, v_{l,1}): 1 \leq l \leq m\} \\ &= d_{G''}(v_{i,k}, v_{i,1}) \\ &= k-1, \\ d_{G''}(z, S_2) &= \min\{d_{G''}(v_{i,k}, v_{l,l+1}): 1 \leq l \leq m\} \\ &= d_{G''}(v_{i,k}, v_{i,i+1}) \\ &= k-i-1. \end{aligned}$$

Therefore, we obtain that

$$d_{G''}(x,S_t)=\begin{cases}0, & \text{if } t=1,\\ 1, & \text{if } t=2,\\ i+1, & \text{if } t=3,\end{cases}$$

$$d_{G''}(y,S_t)=\begin{cases}j-1, & \text{if } t=1,\\ 0, & \text{if } t=2,\\ i+2-j, & \text{if } t=3,\end{cases}$$

$$d_{G''}(z,S_t)=\begin{cases}k-1, & \text{if } t=1,\\ k-i-1, & \text{if } t=2,\\ 0, & \text{if } t=3.\end{cases}$$

By the above notations, we have the following results.

Theorem 3.3. *Let $G' = G \cup E_1 \cup E_2$ and $G \subseteq G'' \subseteq G'$. Then, $pd(G'') = 3$.*

Proof. Since G'' is not a path, $pd(G'') \geq 3$. To show the upper bound of partition dimension of G, define a partition $\Pi = \{S_1, S_2, S_3\}$ of G'' where $S_1 = \{v_{i,1}: 1 \leq i \leq m\}, S_2 = \{v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i+1\}$, and S_3 contains the rest vertices of G. By the definition of partition Π, for a vertex $v_{i,j} \in V(G)$ in S_a where $i \in [1,m]$, $j \in [1,n_i]$ and $a \in [1,3]$, we have the representation of $v_{i,j}$ with respect to the partition Π as follows.

$$r(v_{i,j}|\Pi) = \begin{cases}(0,1,i+1), & \text{if } j=1,\\ (j-1,0,i+2-j), & \text{if } j\in[2,i+1],\\ (j-1,j-i-1,0), & \text{if } j\in[i+2,n_i].\end{cases}$$

Let us show that Π is a resolving partition of G''. We consider any two vertices $x, y \in V(G'')$. If x and $y$are in the different partition class, then clearly that they have distinct representation. Now we suppose that $x, y \in S_a$ for some $a \in [1,3]$. If $x = v_{p,q}$ and $y = v_{p,r}$ where $1 \leq p \leq m$ and $(2 \leq q < r \leq p+1$ or $p+2 \leq q < r \leq n_p)$, then $d_{G''}(x,S_1) = q-1 < r-1 = d_{G''}(y,S_1)$. Therefore, $r(x|\Pi) \neq r(y|\Pi)$.

Now, assume that $x = v_{p,q}$ and $y = v_{r,s}$ in S_a for some $a \in [1,3]$ and $p, r \in [1,m]$ where $p \neq q$. For two vertices $x = v_{p,1}$ and $y = v_{r,1}$ in S_1 where $1 \leq p < r \leq m$, then $d_{G''}(x,S_3) = p+1 < r+1 = d_{G''}(y,S_3)$. For two vertices $x = v_{p,q}$ and $y = v_{r,s}$ in S_2 where $1 \leq p < r \leq m, 2 \leq q \leq p+1$ and $2 \leq s \leq r+1$, if $d_{G''}(x,S_1) = q-1 = s-1 = d_{G''}(y,S_1)$, then $d_{G''}(x,S_3) = p+2-q < r+2-s = d_{G''}(y,S_3)$. Otherwise, $d_{G''}(x,S_1) \neq d_{G''}(y,S_1)$. For two vertices $x = v_{p,q}$ and $y = v_{r,s}$ in S_3 where $1 \leq p < r \leq m$,
$p+2 \leq q \leq n_p$ and $r+2 \leq s \leq n_r$, if $d_{G''}(x,S_1) = q-1 = s-1 = d_{G''}(y,S_1)$, then $d_{G''}(x,S_2) = q-p-1 > r-s-1 = d_{G''}(y,S_3)$. Otherwise, $d_{G''}(x,S_1) \neq d_{G''}(y,S_1)$. Therefore, $(x|\Pi) \neq r(y|\Pi)$ for any two vertices $x = v_{p,q}$ and $y = v_{r,s}$ of $V(G'')$ in S_a for some $a \in [1,3]$.

The four graphs in Figure 4 give an illustration of the graphs provided for Theorem 3.3. These graphs are (a) $G = P_4 \cup P_5 \cup P_6 \cup P_7$, (b) $G' = G \cup E_1 \cup E_2$, (c) $G_1'' = G \cup E_1 \cong G \cup E_2$ and (d) $G_2'' \subset G'$. Note that from Theorem 3.3, $pd(G) = pd(G') = pd(G_1'') = pd(G_2'') = 3$.

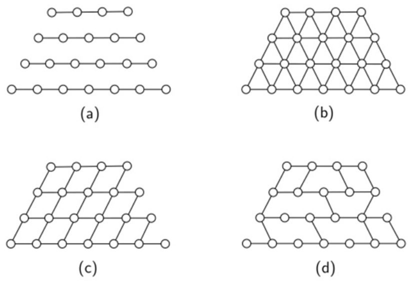

Figure 4: Graphs (a) $G = P_4 \cup P_5 \cup P_6 \cup P_7$, (b) $G' = G \cup E_1 \cup E_2$, (c) $G_1'' = G \cup E_1$, and (d) $G_2'' \subset G'$, where $pd(G) = pd(G') = pd(G_1'') = pd(G_2'') = 3$.

In the next result, we also construct graphs from disjoint union of paths $G = \bigcup_{i=1}^{m} P_{n_i}$, so that their partition dimensions are equal to 3 as well.

Theorem 3.4. *Let $G' = G \cup E_1 \cup E_2, F \subseteq E(G)$ where $F = \{v_{i,j}v_{i,j+1}: 2 \leq i \leq m-1, 1 \leq j \leq n_i - 1\}$ and $F' \subseteq F$. Then, $pd(G' - F') = 3$.*

Proof. Let $H = G' - F'$. This is easy to see that $pd(H) \geq 3$. To show that $pd(H) \leq 3$, define a partition $\Pi' = \{S_1', S_2', S_3'\}$ of H where $S_1' = \{v_{i,1}: 1 \leq i \leq m\}, S_2' = \{v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i+1\}$ and $S_3' = \{v_{i,j}: 1 \leq i \leq m, i+2 \leq j \leq n_i\}$.

By the definition of a partition Π' of $V(H)$, for three vertices $x, y, z \in V(H)$ where $x = v_{i,1} \in S_1', y = v_{i,j} \in S_2'$ and $z = v_{i,k} \in S_3'$ for some $i \in [2,m], j \in [2,i+1]$ and $k \in [i+2, n_i]$, we have

$$d_H(x,S_2') = \min\{d_H(v_{i,1},v_{l,2}): 1 \leq l \leq m\}$$
$$= d_H(v_{i,1},v_{i+1,2})$$
$$= 1,$$

$d_H(x, S_3') = \min\{d_H(v_{i,1}, v_{l,l+2}): 1 \le l \le m\}$
$= d_H(v_{i,1}, v_{1,3})$
$= i + 1,$
$d_H(y, S_1') = \min\{d_H(v_{i,j}, v_{l,1}): 1 \le l \le m\}$
$= \begin{cases} d_H(v_{i,j}, v_{i-j+1,1}), & \text{if } j \le i, \\ d_H(v_{i,j}, v_{1,1}), & \text{if } j = i + 1. \end{cases}$
$= \begin{cases} j - 1, & \text{if } j \le i, \\ i, & \text{if } j = i + 1. \end{cases}$
$= j - 1$
$d_H(y, S_3') = \min\{d_H(v_{i,j}, v_{l,l+2}): 1 \le l \le m\}$
$= \begin{cases} d_H(v_{i,j}, v_{1,3}), & \text{if } j = 2, \\ d_H(v_{i,j}, v_{j-2,j}), & \text{if } j \ne 2. \end{cases}$
$= \begin{cases} i, & \text{if } j = 2, \\ i - j + 2 & \text{if } j \ne 2. \end{cases}$
$= i + 2 - j$
$d_H(z, S_1') = \min\{d_H(v_{i,k}, v_{l,1}): 1 \le l \le m\}$
$= d_H(v_{i,k}, v_{1,1})$
$= d_H(v_{i,k}, v_{1,k-i+1}) + d_H(v_{1,k-i+1}, v_{1,1})$
$= k - 1$
$d_H(z, S_2') = \min\{d_H(v_{i,k}, v_{l,l+1}): 1 \le l \le m\}$
$= \begin{cases} d_H(v_{i,k}, v_{k-1,k}), & \text{if } k \le m - 1, \\ d_H(v_{i,k}, v_{m,m+1}), & \text{if } k > m - 1. \end{cases}$
$= \begin{cases} k - 1 - i, & \text{if } k \le m - 1, \\ (m - i) + (k - m - 1) & \text{if } k > m - 1. \end{cases}$
$= k - 1 - i.$
.

By considering the resolving $3-$partition $\Pi = \{S_1, S_2, S_3\}$ of a graph G' in Theorem 3.3, we obtain that for any vertex $x \in V(G')$ where $V(G') = V(H)$, then $r(x|\Pi) = r(x|\Pi')$. Therefore, $r(x|\Pi') \ne r(y|\Pi')$for any two vertices $x, y \in V(H)$ and so that Π' is a resolving partition of H.

Figure 5 represents some graphs satisfying Theorem 3.4, namely (a) $H = G' - F$, (b) $H_1 \supset H$ and (c) $H_2 \supset H$ where $H_1 = G' - F_1'$ and $H_2 = G' - F_2'$ for some $F_1', F_2' \subseteq F$. Note that from Theorem 3.4, $pd(H) = pd(H_1') = pd(H_2') = 3$.

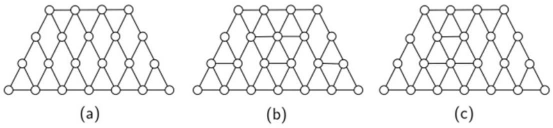

Figure 5: Graphs (a) $H = G' - F$, (b) $H_1 \supset H$ and (c) $H_2 \supset H$ where $H_1 = G' - F_1'$ and $H_2 = G' - F_2'$ for some $F_1', F_2' \subseteq F$.

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REFERENCES

Asmiati and Baskoro, E. T. (2012). Characterizing all graphs containing cycle with the locating-chromatic number 3. *AIP Conference Proceedings,* 1450:351–357.

Baskoro, E. T. and Asmiati (2013). Characterizing all trees with locating-chromatic number 3. *Electronic Journal of Graph Theory and Applications*, 1(2):109–117.

Chartrand, G., Erwin, D., Henning, M. A., Slater, P. J., and Zhang, P. (2002). The locating-chromatics number of a graph. *Bulletin of the Institute of Combinatorics and its Applications*, 36:89–101.

Chartrand, G., Erwin, D., Henning, M. A., Slater, P. J., and Zhang, P. (2003). Graph of order n with locatingchromatics number n − 1. *Discrete Mathematics*, 269:65–79.

Chartrand, G., Salehi, E., and Zhang, P. (1998). On the partition dimension of a graph. *Congressus Numerantium*, 130:157–168.

Chartrand, G., Salehi, E., and Zhang, P. (2000). The partition dimension of a graph. *Aequationes Mathematicae*, 59:45–54.

Haryeni, D. O., Baskoro, E. T., and Saputro, S. W. (2017). On the partition dimension of disconnected graphs. *Journal of Mathematical and Fundamental Sciences*, 49(1):18–32.

Haryeni, D. O., Baskoro, E. T., and Saputro, S. W. (2019). A method to construct graphs with certain partition dimension. *Electronic Journal of Graph Theory and Applications*, 7(2):251–263.

Rodr´ıguez-Velazquez, J. A., Yero, I. G., and Kuziak, D. ´ (2016). The partition dimension of corona product graphs. *Ars Combinatoria*, 127:387–399.

Tomescu, I. (2008). Discrepancies between metric dimension and partition dimension of a connected graph. *Discrete Mathematics*, 308:5026–5031.

Welyyanti, D., Baskoro, E. T., Simanjuntak, R., and Uttunggadewa, S. (2014). The locating-chromatic number of disconnected graphs. *Far East Journal of Mathematical Sciences*, 94(2):169–182.

Yero, I. G., Jakovac, M., Kuziak, D., and Taranenko, A. (2014). The partition dimension of strong product graphs and cartesian product graphs. *Discrete Mathematics*, 331:43–52.

Yero, I. G., Kuziak, D., and Rodr´ıguez-Velazquez, J. A. ´ (2010). A note on the partition dimension of cartesian product graphs. *Applied Mathematics and Computation*, 217:3571–3574.