Hermite Interpolation by Piecewise Cubic Trigonometric Spline with Shape Parameters

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Abstract: In this paper, we are studing in depth a new cubic Hermite trigonometric spline interpolation method for curves and surfaces with shape parameters. Based on this model of interpolation, we will give some examples of free-form curves and surfaces, and analyse the effect of different shape parameters on the curve and surface shape. We show that for $\lambda = 1-\sqrt{2}$ and $\alpha = \frac{1}{4}$ the obtained cubic Hermite trigonometric curve or surface is C^3 continuous. Finally, we will give an example of application, and we will discuss how the adjustment of the shape parameters affects the shape of freeform surfaces.

1 INTRODUCTION

Nowadays, the worldwide trend of the mechanic industry and more exactly the automotive industry and aviation is marketed depend not only on functional requirements, but also aesthetic ones. In addition, the continuous increase in energy charge is pushing manufacturers to design product with aerodynamic,functional and aesthetic freeform shapes (Savio et al., 2007). The design and prototyping stage of free-form parts require the use of real model.

In the world of the automotive industry for example, the initial design of a car wing is often done by designers which concretize their concept by producing a model part. To begin or continue the production process from these real models. They must be transferred to a CAD system as a CAD model (Hajji et al., 2018). Since this process aims to create a CAD model from a physical part, (reverse engineering) and obtain a good CAD model, the most important step is the choice of the model to approximate the complex surface from the set of points measured on the real object.

The geometry of the curves and surfaces are two theories which make it possible to describe the complex forms. These complex shapes are usually described using parametric surface representations (Farin, 2001). The commonly used parametric surfaces are Ferguson et Coons Hermite, Spline, Bezier, B-spline and B-spline surfaces, NURBS. Two excellent references are (Farin, 2001), (Farin, 1999).

The paper is arranged as follows. Section 2, recalls the definition of cubic trigonometric polynomial B-spline basis function (see (Liu et al., 2012)). In Section 3, the cubic trigonometric polynomial B-spline curve is given. Section 4 describes the construction of cubic Hermite trigonometric spline interpolation, which is based on determining a set of cubic Hermite trigonometric B-splines functions with shape parameters. We will also give the definition of cubic Hermite trigonometric spline curve associated at this construction. Section 5 deals with the definition and the smoothness of the interpolating surfaces. When the shape parameters satisfy a simple condition, the interpolating surface is C^3. Finally, in order to illustrate our results, we will give in Section 6 some numerical examples.

2 CUBIC TRIGONOMETRIC POLYNOMIAL B-SPLINE BASIS FUNCTION

In this section, we recall the definition and the interesting properties of cubic trigonometric polynomial B-spline Basis function, for more details see (Liu et al., 2012).

Definition 1. *For shape parameter, where* $-1 \leq \lambda \leq 1$, $t \in [0, \frac{\pi}{2}]$, *the following four functions are defined*

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as the cubic trigonometric B-spline basis function with a shape parameters λ:

$$\begin{cases} B_0(\lambda,t) = f(\lambda)(1-\sin(t))(1-\lambda\sin(t))^2, \\ B_1(\lambda,t) = f(\lambda)(1+\cos(t))(1+\lambda\cos(t))^2, \\ B_2(\lambda,t) = f(\lambda)(1+\sin(t))(1+\lambda\sin(t))^2, \\ B_3(\lambda,t) = f(\lambda)(1-\cos(t))(1-\lambda\cos(t))^2, \end{cases} \quad (1)$$

where $f(\lambda) = \frac{1}{2\lambda^2+4\lambda+4}$.

The cubic trigonometric polynomial B-spline Basis function possesses all the desirable properties of classical polynomial B-splines, see (Liu et al., 2012).

- Nonnegative and Partition of unity: $B_i(\lambda,t) \geq 0,\ i = 0,1,2,3$ and $\sum_{i=0}^{3} B_i(\lambda,t) = 1$;
- Symmetry: $B_0(\lambda,t) = B_3(\lambda,\frac{\pi}{2}-t),\ B_1(\lambda,t) = B_2(\lambda,\frac{\pi}{2}-t)$;
- Monotony: where $t \in [0,\frac{\pi}{2}]$, $B_0(\lambda,t)$ and $B_3(\lambda,t)$ are monotonically decreasing for shape parameter λ. $B_1(\lambda,t)$ and $B_2(\lambda,t)$ are monotonically increasing for shape parameters λ, respectively.

3 CUBIC TRIGONOMETRIC B-SPLINE CURVE WITH A SHAPE PARAMETER

3.1 Definition and Properties

Definition 2. *Given points* $V_i (i = 0,1,\cdots,n+1)$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $-1 \leq \lambda \leq 1$ *and knots vectors* $U = [u_1,u_2,\cdots,u_n]$. *For* $i = 1,\ldots,n-1$, *the* i^{th} *trigonometric curve segment is given by:*

$$\mathcal{V}_i(\lambda,t) := \sum_{j=0}^{3} V_{i+j-1}\, B_j(\lambda,t), \quad t \in [0,\tfrac{\pi}{2}]. \quad (2)$$

In the same way, we can define the cubic trigonometric polynomial B-spline curve as follows :

$$\mathcal{V}(\lambda,t) := \mathcal{V}_i(\lambda,\tfrac{\pi}{2}.\tfrac{t-u_i}{\Delta u_i}), \quad t \in [u_i,u_{i+1}], \quad (3)$$

where $\Delta u_i = u_{i+1} - u_i$, $i = 1,2,\cdots,n-1$, *U is equidistant knots vectors.*

The cubic trigonometric B-spline curve (2) has the following important geometric properties. For more details see ((Liu et al., 2012)):

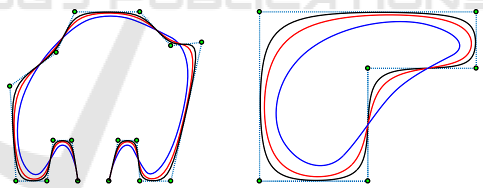
1. Terminal properties:

$$\begin{cases} \mathcal{V}(\lambda,0) = \frac{V_0-2V_1+V_2}{2\lambda^2+4\lambda+4} + V_1, \\ \mathcal{V}(\lambda,\frac{\pi}{2}) = \frac{V_1-2V_2+V_3}{2\lambda^2+4\lambda+4} + V_2, \\ \mathcal{V}'(\lambda,0) = \frac{(2\lambda+1)(V_2-V_0)}{2(\lambda(\lambda+2)+2)}, \\ \mathcal{V}'(\lambda,\frac{\pi}{2}) = \frac{(2\lambda+1)(V_3-V_1)}{2(\lambda(\lambda+2)+2)}. \end{cases} \quad (4)$$

2. Symmetry: $V_0,\cdots,V_3$ and $V_3,\cdots,V_0$ define the same trigonometric B-spline curve, i.e., $\mathcal{V}(\lambda,t) = \mathcal{V}(\lambda,\frac{\pi}{2}-t)$.
3. Geometric invariance: since the blending functions have the properties of partition of unity, the shape of these trigonometric B-spline curves is independent of the choice of coordinates.
4. Convex hull property: the blending functions have the properties of nonnegativity and partition of unity, as a consequence, the entire trigonometric B-spline curve segment must lie inside the control polygon spanned by $V_0,\cdots,V_3$.
5. Variation diminishing property.

3.2 Numerical Examples

By using the terminal properties we can construct an open curve $\mathcal{V}(\lambda,t)$ interpolating V_0 and V_{n+1}. Indeed, it suffices to add four control points $V_{-2} = V_{-1} = V_0$ and $V_{n+3} = V_{n+2} = V_{n+1}$. For constructing a trigonometric closed curve $\mathcal{V}(\lambda,t)$, we add four control points $V_{-2} = V_n$, $V_{-1} = V_{n+1}$, $V_{n+2} = V_0$ and $V_{n+3} = V_1$. In Figure 1, the open and closed curves are generated by altering the value of $\lambda = 0.5$: blue color, $\lambda = 0.8$: red color and $\lambda = 1$: black color, U is equidistant knots vectors. As λ increases, the curve is closer to the control polygon.



(a) (b)

Figure 1: Effect of varying the shape parameter λ on the curve.

4 CUBIC HERMITE TRIGONOMETRIC SPLINE INTERPOLATION

4.1 Basis of the Cubic Hermite Trigonometric Spline Interpolation

In analogy with the classical cubic polynomial Hermite function basis, we shall determine our trigonometric Hermite basis $TB_0^\alpha(t)$, $TB_1^\alpha(t)$, $TB_2^\alpha(t)$ and

$TB_3^\alpha(t)$ first by focusing on the interval $[0, \frac{\pi}{2}]$ and imposing the four required boundary (endpoints) conditions in each case, i.e.

$$TB_0^\alpha(0)=0, TB_0^\alpha(\tfrac{\pi}{2})=0, TB_0^{\alpha(1)}(0)=-\alpha_1, TB_0^{\alpha(1)}(\tfrac{\pi}{2})=0,$$
$$TB_1^\alpha(0)=1, TB_1^\alpha(\tfrac{\pi}{2})=0, TB_1^{\alpha(1)}(0)=0, TB_1^{\alpha(1)}(\tfrac{\pi}{2})=-\alpha_2,$$
$$TB_2^\alpha(0)=0, TB_2^\alpha(\tfrac{\pi}{2})=1, TB_2^{\alpha(1)}(0)=\alpha_1, TB_2^{\alpha(1)}(\tfrac{\pi}{2})=0,$$
$$TB_3^\alpha(0)=0, TB_3^\alpha(\tfrac{\pi}{2})=0, TB_3^{\alpha(1)}(0)=0, TB_3^{\alpha(1)}(\tfrac{\pi}{2})=\alpha_2.$$

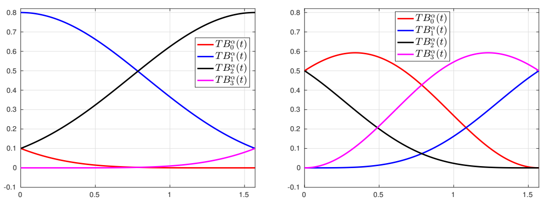
The existence of the functions TB_0^α, TB_1^α, TB_2^α and TB_3^α is guaranteed if we consider $\langle B_0(\lambda,t), B_1(\lambda,t), B_2(\lambda,t), B_3(\lambda,t)\rangle$ as solutions space. With a simple calculation we obtain the following expressions:

$$\begin{pmatrix} TB_0^\alpha(\lambda,t) \\ TB_1^\alpha(\lambda,t) \\ TB_2^\alpha(\lambda,t) \\ TB_3^\alpha(\lambda,t) \end{pmatrix} = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_1 \\ \gamma_1 & \gamma_2 & \gamma_1 & \gamma_0 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_1 \\ \delta_1 & \delta_2 & \delta_1 & \delta_0 \end{pmatrix} \begin{pmatrix} B_0(\lambda,t) \\ B_1(\lambda,t) \\ B_2(\lambda,t) \\ B_3(\lambda,t) \end{pmatrix} \tag{5}$$

with

$$\delta_0 = \frac{\alpha(2\lambda(\lambda+2)(\lambda(\lambda+2)+2)+1)}{\lambda(\lambda+2)(2\lambda+1)},$$
$$\delta_1 = \frac{-\alpha(\lambda+1)^2}{\lambda(\lambda+2)(2\lambda+1)},$$
$$\delta_2 = \frac{\alpha}{\lambda(\lambda+2)(2\lambda+1)},$$
$$\gamma_0 = \frac{(\lambda+1)^2(2\lambda+1)-\alpha(2\lambda(\lambda+2)(\lambda(\lambda+2)+2)+1)}{\lambda(\lambda+2)(2\lambda+1)},$$
$$\gamma_1 = \frac{\alpha(\lambda+1)^2-2\lambda-1}{\lambda(\lambda+2)(2\lambda+1)},$$
$$\gamma_2 = \frac{(\lambda+1)^2(2\lambda+1)-\alpha}{\lambda(\lambda+2)(2\lambda+1)}.$$

The four trigonometric Hermite basis functions TB_i^α, $i=0,\ldots,3$ are illustrated graphically in Figure 2. The trigonometric Hermite B-splines TB_i^α, $i=0,\ldots,3$, has the following properties.



(a) $\lambda=1$ and $\alpha=0.2$. (b) $\lambda=-1$ and $\alpha=0.7$.

Figure 2: Trigonometric Hermite basis functions TB_i^α, $i=0,\ldots,3$, with various choice of λ and α.

Proposition 3. *Let $-1\le\lambda\le1$. For any positive integer α, the trigonometric functions TB_i^α, $i=0,\cdots,3$ satisfy the following properties:*

- *Partition of unity:* $\sum_{i=0}^{3} TB_i^\alpha(\lambda,t)=1, \ t\in[0,\frac{\pi}{2}]$.
- *Symmetry:* $TB_0^\alpha(\lambda,t)=TB_3^\alpha(\lambda,\frac{\pi}{2}-t), TB_1^\alpha(\lambda,t)=TB_2^\alpha(\lambda,\frac{\pi}{2}-t)$.

Proof. Indeed

$$\sum_{i=0}^{3} TB_i^\alpha(\lambda,t) = (\delta_0+\gamma_1+\gamma_0+\delta_1)(B_0(\lambda,t)+B_3(\lambda,t)) + (\delta_1+\gamma_2+\gamma_1+\delta_2)(B_1(\lambda,t)+B_2(\lambda,t)).$$

On the other hand it is easy to verify that, $(\delta_0+\gamma_1+\gamma_0+\delta_1)=1$ and $(\delta_1+\gamma_2+\gamma_1+\delta_2)=1$. Then

$$\sum_{i=0}^{3} TB_i^\alpha(\lambda,.) = B_0(\lambda,.)+B_1(\lambda,.)+B_2(\lambda,.)+B_3(\lambda,.)=1.$$

The symmetry stems from the symmetry of the basis functions $B_i(\lambda,t)$ and the symmetry of lines 1 and 4 (respectively 2 and 3) of the system matrix (5).

4.2 Cubic Hermite Trigonometric Spline Interpolation with Shape Parameters

Suppose that we are given four distinct points $P_j\in\mathbb{R}^2$, $j=0,\cdots,3$. We are looking for a solution of the trigonometric Hermite interpolation problem

$$\begin{cases} TH_\alpha(\lambda,0)=P_1, \\ TH_\alpha(\lambda,\frac{\pi}{2})=P_2, \\ TH_\alpha^{(1)}(\lambda,0)=\alpha(P_2-P_0), \\ TH_\alpha^{(1)}(\lambda,\frac{\pi}{2})=\alpha(P_3-P_1). \end{cases} \tag{6}$$

where $TH_\alpha(\lambda,t):[0,\frac{\pi}{2}]\longrightarrow\mathbb{R}^2$ is a cubic parametric trigonometric curve, α is positive real and $-1\le\lambda\le1$. This situation is illustrated in Figure 3 (a). In Figure 3 (b), we drew the cubic trigonometric Hermite spline curves associated with the four points $P_j\in\mathbb{R}^2$, $j=0,\cdots,3$, with various choice of shape parameters λ and α.

Proposition 4. *Let $-1\le\lambda\le1$ and α positive reel. For any interpolation points $P_j, j=0,\cdots,3$, there exists a unique Hermite trigonometric spline interpolation with shape parameters*

$$TH_\alpha(\lambda,t)=\sum_{j=0}^{3} P_j\, TB_j^\alpha(\lambda,t),\ t\in[0,\tfrac{\pi}{2}],$$

satisfying the interpolation conditions (6).

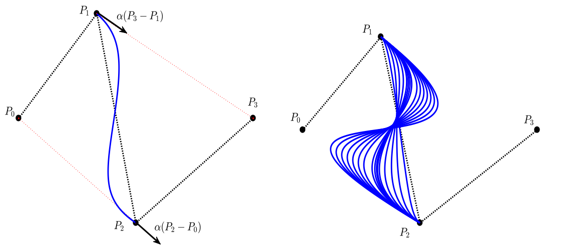
4.3 Hermite Trigonometric Spline Curve

Definition 5. *Given points $P_i (k=0,1,\cdots,n+1)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, $-1\le\lambda\le1$, $\alpha>0$ and knots vectors $U=[u_1,u_2,\cdots,u_n]$. For $i=1,\ldots,n-1$, the i^{th} Hermite trigonometric spline curve segment is given by:*

$$\mathcal{P}_i(\lambda,t) := \sum_{j=0}^{3} P_{i+j-1}\, TB_j^\alpha(\lambda,t),\ \ t\in[0,\tfrac{\pi}{2}]. \tag{7}$$

In the same way, we can define the Hermite trigonometric spline curve made by all segments as:

$$\mathcal{P}(\lambda,t) := \mathcal{P}_i(\lambda,\tfrac{\pi}{2}\cdot\tfrac{t-u_i}{\Delta u_i}),\ \ t\in[u_i,u_{i+1}], \tag{8}$$



(a) (b)

Figure 3: (a) Cubic trigonometric Hermite spline curve created with four control points. (b) Cubic trigonometric Hermite spline curves with various choices of λ and α..

where $\Delta u_i = u_{i+1} - u_i$, $i = 1,2,\cdots,n-1$, *U is equidistant knots vectors.*

Lemma 6. *For Hermite trigonometric spline curve (8), its continuity is as follows:*

$$\mathcal{P}^{(k)}(\lambda,u_i^-) = \left(\frac{\Delta u_i}{\Delta u_{i+1}}\right)^k \mathcal{P}^{(k)}(\lambda,u_i^+), \qquad (9)$$

(a) *when* $\lambda = 1-\sqrt{2}$ *and* $\alpha = \frac{1}{4}$, $k = 0,1,2,3$.

(b) *when* $\lambda \neq 1-\sqrt{2}$ *and* $\forall\, \alpha > 0$, $k = 0,1,3$.

Proof.For (7), according to simple differential operation, until to third derivation, and more calculate, we can gain:

$$\begin{cases} \mathcal{P}_{i-1}(\lambda,\frac{\pi}{2}) = \mathcal{P}_i(\lambda,0) = P_i \\ \mathcal{P}^{(1)}_{i-1}(\lambda,\frac{\pi}{2}) = \mathcal{P}^{(1)}_i(\lambda,0) = \alpha(P_{i+1}-P_{i-1}) \\ \mathcal{P}^{(2)}_{i-1}(\lambda,\frac{\pi}{2}) = \frac{\alpha(2(\lambda+1)^2P_{i+1}+((\lambda-2)\lambda-1)(P_{i-2}-P_i))}{2\lambda+1} \\ \qquad + \frac{2(-\alpha(\lambda+1)^2+2\lambda+1)P_{i-1}-2(2\lambda+1)P_i}{2\lambda+1} \\ \mathcal{P}^{(2)}_i(\lambda,0) = \frac{\alpha(2(\lambda+1)^2(P_{i-1}-P_{i+1})+((\lambda-2)\lambda-1)P_{i+2})}{2\lambda+1} \\ \qquad + \frac{(\alpha(1-(\lambda-2)\lambda)-4\lambda-2)P_i+2(2\lambda+1)P_{i+1}}{2\lambda+1} \\ \mathcal{P}^{(3)}_{i-1}(\lambda,\frac{\pi}{2}) = \mathcal{P}^{(3)}_i(\lambda,0) = \frac{5}{4}(P_{i+1}-P_{i-1}). \end{cases}$$

So that,

$$\mathcal{P}^{(k)}_{i-1}(\lambda,\frac{\pi}{2}) = \mathcal{P}^{(k)}_i(\lambda,0),\ |\lambda| \le 1,\ \alpha > 0,\ k = 0,1,3. \quad (10)$$

Let $u \in [u_i,\ u_{i+1}]$, $t = \frac{\pi}{2}.\frac{u-u_i}{\Delta u_i}$, then $\mathcal{P}^{(k)}(\lambda,u) = \left(\frac{\pi}{2}.\frac{1}{\Delta u_i}\right)^k \mathcal{P}^{(k)}_i(\lambda,t)$.

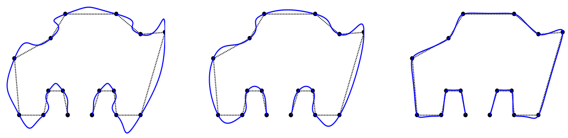
Then

$$\mathcal{P}^{(k)}(\lambda,u_i^-) = \left(\frac{\pi}{2}.\frac{1}{\Delta u_{i-1}}\right)^k \mathcal{P}^{(k)}_{i-1}(\lambda,\frac{\pi}{2}), \qquad (11)$$

$$\mathcal{P}^{(k)}(\lambda,u_i^+) = \left(\frac{\pi}{2}.\frac{1}{\Delta u_i}\right)^k \mathcal{P}^{(k)}_{i-1}(\lambda,0).$$

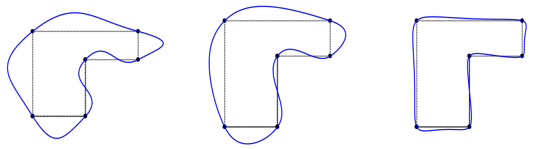
According to (10) and (11), the Lemma 6 holds. The Lemma 6 shows that $\mathcal{P}(\lambda,u)$ is $\mathcal{C}^1$ continuity if $\lambda = 1-\sqrt{2}$ and $\alpha = \frac{1}{4}$, is $\mathcal{C}^3$ continuity.

In Figures 4 and 5, we give an examples of open and close cubic trigonometric Hermite spline interpolant curves with various choice of λ and α. In the case $\lambda = 1-\sqrt{2}$, $\alpha = \frac{1}{4}$ the curve is is $\mathcal{C}^3$ continuity, on the other hand in the other cases we only have a is $\mathcal{C}^1$ continuity.



(a) $\lambda=-0.1, \alpha=\frac{3}{4}$. (b) $\lambda=0.5, \alpha=\frac{1}{2}$. (c) $\lambda=1-\sqrt{2}, \alpha=\frac{1}{4}$.

Figure 4: An open cubic trigonometric Hermite spline interpolant curves with various choice of λ and α.



(a) $\lambda=-0.1, \alpha=\frac{3}{4}$. (b) $\lambda=0.5, \alpha=\frac{1}{2}$. (c) $\lambda=1-\sqrt{2}, \alpha=\frac{1}{4}$.

Figure 5: A close cubic trigonometric Hermite spline interpolant curves with various choice of λ and α.

5 CUBIC TRIGONOMETRIC HERMITE PARAMETRIC SPLINE SURFACES

Similarly to the work done by Y. ZHU et al. (see (Zhu et al., 2012)), we define trigonometric parametric spline surfaces as a tensor product. More precisely we have the following definition.

Definition 7. *Given* $(n+2)\times(n+2)$ *interpolation points* P_{ij}, *knot vectors* $U = [u_1,u_2,\cdots,u_n]$ *and* $\alpha > 0$. *For* $-1 \le \lambda \le 1$, *the trigonometric parametric spline surface patch has the form :*

$$\mathcal{S}_{i,j}(\lambda,t,s) := \sum_{k=0}^{3}\sum_{l=0}^{3} P_{i+k-1,j+l-1}\, TB_k^{\alpha}(\lambda,t)TB_l^{\alpha}(\lambda,s),$$
$$(t,s) \in [0,\tfrac{\pi}{2}]\times[0,\tfrac{\pi}{2}].$$

Then the trigonometric parametric spline surface is given by,

$$\mathcal{S}(\lambda,t,s) := \mathcal{S}_{i,j}(\lambda,\tfrac{\pi}{2}.\tfrac{t-u_i}{\Delta u_i},\tfrac{\pi}{2}.\tfrac{s-u_j}{\Delta u_j}), \qquad (12)$$
$$(t,s) \in [u_i,u_{i+1}]\times[u_j,u_{j+1}].$$

where $\Delta u_i = u_{i+1} - u_i$.

The surface $\mathcal{S}(\lambda,t,s)$ has the following interpolation property.

Lemma 8. *The $(i,j)^{th}$ bi-trigonometric Hermite segment patch $S_{i,j}(u,v)$ verifies the following interpolating properties:*

$$\begin{aligned}\mathcal{S}_{i,j}(\lambda,0,0) &= \mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})=P_{i,j},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial}{\partial t}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial}{\partial t}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \tfrac{\partial}{\partial t}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\tfrac{\partial}{\partial t}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})\\ &= \alpha(P_{i+1,j}-P_{i-1,j}),\end{aligned}$$

$$\begin{aligned}\tfrac{\partial}{\partial s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial}{\partial s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \tfrac{\partial}{\partial s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\tfrac{\partial}{\partial s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})\\ &= \alpha(P_{i,j+1}-P_{i,j-1}),\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial t\partial s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial^2}{\partial t\partial s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \tfrac{\partial^2}{\partial t\partial s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\tfrac{\partial^2}{\partial t\partial s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})\\ &= \alpha^2(P_{i-1,j-1}-P_{i-1,j+1}-P_{i+1,j-1}+P_{i+1,j+1}),\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 t}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i-1,j}}{2\lambda+1}-\\ &\quad \tfrac{2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i+1,j}+\alpha((\lambda-2)\lambda-1)P_{i+2,j}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 t}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2}) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i-1,j}}{2\lambda+1}-\\ &\quad \tfrac{2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i+1,j}+\alpha((\lambda-2)\lambda-1)P_{i+2,j}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 t}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i+1,j}}{2\lambda+1}+\\ &\quad \tfrac{\alpha((\lambda-2)\lambda-1)P_{i-2,j}-2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i-1,j}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 t}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2}) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i+1,j}}{2\lambda+1}+\\ &\quad \tfrac{\alpha((\lambda-2)\lambda-1)P_{i-2,j}-2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i-1,j}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i,j-1}}{2\lambda+1}-\\ &\quad \tfrac{2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i,j+1}+\alpha((\lambda-2)\lambda-1)P_{i,j+2}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2} &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i,j+1}}{2\lambda+1}+\\ &\quad \tfrac{\alpha((\lambda-2)\lambda-1)P_{i,j-2}-2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i,j-1}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i,j+1}}{2\lambda+1}+\\ &\quad \tfrac{\alpha((\lambda-2)\lambda-1)P_{i,j-2}-2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i,j-1}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^2}{\partial^2 s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2}) &= \tfrac{\left(-\alpha\lambda^2+2(\alpha-2)\lambda+\alpha-2\right)P_{i,j}+2\alpha(\lambda+1)^2P_{i,j+1}}{2\lambda+1}+\\ &\quad \tfrac{\alpha((\lambda-2)\lambda-1)P_{i,j-2}-2\left(\alpha(\lambda+1)^2-2\lambda-1\right)P_{i,j-1}}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^3}{\partial^3 t}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial^3}{\partial^3 t}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})=\tfrac{\partial^3}{\partial^3 t}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)\\ &= \tfrac{\partial^3}{\partial^3 t}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})=-\tfrac{\alpha\left(6\lambda^2-2\lambda-1\right)(P_{i-1,j}-P_{i+1,j})}{2\lambda+1},\end{aligned}$$

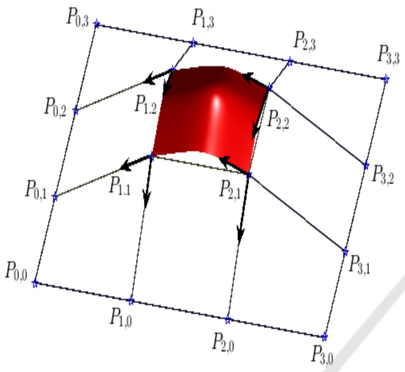
$$\begin{aligned}\tfrac{\partial^3}{\partial^3 s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial^3}{\partial^3 s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})=\tfrac{\partial^3}{\partial^3 s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)\\ &= \tfrac{\partial^3}{\partial^3 s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})=-\tfrac{\alpha\left(6\lambda^2-2\lambda-1\right)(P_{i,j-1}-P_{i,j+1})}{2\lambda+1},\end{aligned}$$

$$\begin{aligned}\tfrac{\partial^3}{\partial^2 t\partial s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial^3}{\partial^2 t\partial s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \tfrac{\partial^3}{\partial^2 t\partial s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\tfrac{\partial^3}{\partial^2 t\partial s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})\\ &= \Big(\alpha\lambda^2P_{i,j-1}-\alpha\lambda^2P_{i,j+1}+2\alpha\lambda^2P_{i+1,j-1}-\\ &\quad 2\alpha\lambda^2P_{i+1,j+1}-\alpha\lambda^2P_{i+2,j-1}+\alpha\lambda^2P_{i+2,j+1}\\ &\quad -2\alpha\lambda P_{i,j-1}+2\alpha\lambda P_{i,j+1}+4\alpha\lambda P_{i+1,j-1}-\\ &\quad 4\alpha\lambda P_{i+1,j+1}+2\alpha\lambda P_{i+2,j-1}-2\alpha\lambda P_{i+2,j+1}\\ &\quad -2\alpha(\lambda+1)^2P_{i-1,j-1}+2\alpha(\lambda+1)^2P_{i-1,j+1}-\\ &\quad \alpha P_{i,j-1}+\alpha P_{i,j+1}+2\alpha P_{i+1,j-1}-2\alpha P_{i+1,j+1}\\ &\quad +\alpha P_{i+2,j-1}-\alpha P_{i+2,j+1}+4\lambda P_{i,j-1}-4\lambda P_{i,j+1}\\ &\quad -4\lambda P_{i+1,j-1}+4\lambda P_{i+1,j+1}+2P_{i,j-1}\\ &\quad -2P_{i,j+1}-2P_{i+1,j-1}+2P_{i+1,j+1}\Big)\tfrac{\alpha}{1+\lambda},\end{aligned}$$

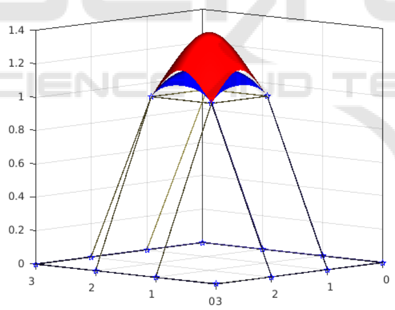
$$\begin{aligned}\tfrac{\partial^3}{\partial t\partial^2 s}\mathcal{S}_{i,j}(\lambda,0,0) &= \tfrac{\partial^3}{\partial t\partial^2 s}\mathcal{S}_{i,j-1}(\lambda,0,\tfrac{\pi}{2})\\ &= \tfrac{\partial^3}{\partial t\partial^2 s}\mathcal{S}_{i-1,j}(\lambda,\tfrac{\pi}{2},0)=\tfrac{\partial^3}{\partial t\partial^2 s}\mathcal{S}_{i-1,j-1}(\lambda,\tfrac{\pi}{2},\tfrac{\pi}{2})\\ &= \Big(2\alpha\lambda^2P_{i-1,j+1}-\alpha\lambda^2P_{i-1,j+2}+2\alpha\lambda^2P_{i+1,j-1}-\\ &\quad \alpha\lambda^2P_{i+1,j}-2\alpha\lambda^2P_{i+1,j+1}+4\alpha\lambda P_{i-1,j+1}\\ &\quad 2\alpha\lambda P_{i-1,j+2}+4\alpha\lambda P_{i+1,j-1}+2\alpha\lambda P_{i+1,j}-\\ &\quad +4\alpha\lambda P_{i+1,j+1}-2\alpha(\lambda+1)^2P_{i-1,j-1}+(\alpha(\lambda-2)\\ &\quad 4\lambda+2)P_{i-1,j}+\alpha((\lambda-2)\lambda-1)P_{i+1,j+2}\\ &\quad +\alpha P_{i-1,j+2}+2\alpha P_{i+1,j-1}+\alpha P_{i+1,j}-2\alpha P_{i+1,j+1}\\ &\quad -4\lambda P_{i-1,j+1}-4\lambda P_{i+1,j}+4\lambda P_{i+1,j+1}-\\ &\quad 2(P_{i-1,j+1}+P_{i+1,j}-P_{i+1,j+1})\Big)\tfrac{\alpha}{1+\lambda}.\end{aligned}$$

Figure 6(a), illustrate the data used to determine the bi-trigonometric Hermite segment patch $S_{i,j}(u,v)$.

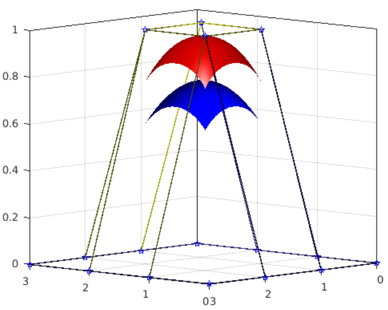
Figure 6(b), shows bi-trigonometric Hermite surface patch with different parameters: $\lambda = -1$ and $\alpha = 0.2$ (red) and $\lambda = 1$ and $\alpha = 0.2$ (blue). Figure 6(c), shows bicubic trigonometric surface patch using the definition of H. Liu et al. (see (Liu et al., 2012)) with different parameters: $\lambda = 1$ (red) and $\lambda = 0$ (blue).



(a)



(b)



(c)

Figure 6: Bicubic trigonometric and bi-trigonometric Hermite B-spline surface patch with different parameters.

The surface $\mathcal{S}(\lambda, t, s)$ resulting of union of bi-trigonometric Hermite segments patch $\mathcal{S}_{i,j}(\lambda, t, s)$ will have continuous first order derivatives and continuous crossed derivative. In the cubic trigonometric Hermite surfaces $\mathcal{S}(\lambda, t, s)$, $\mathcal{C}^1$ continuity between two surfaces is a direct consequence of the construction of this interpolant. So much so the $\mathcal{C}^3$ continuity is forced by imposing $\lambda = 1 - \sqrt{2}$ and $\alpha = \frac{1}{4}$. More precisely, we have the following result.

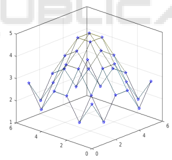
Proposition 9. *The surface (12) is*

(a) $\mathcal{C}^3$*-continuous, when* $\lambda = 1 - \sqrt{2}$ *and* $\alpha = \frac{1}{4}$.

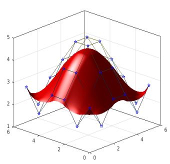
(b) $\mathcal{C}^1$*-continuous, when* $\lambda \neq 1 - \sqrt{2}$ *and* $\forall\, \alpha > 0$.

6 EXAMPLES FOR APPLICATION

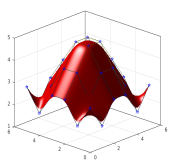
In order to justify the accuracy and efficiency of our presented cubic trigonometric Hermite interpolation we consider some graphical examples. For constructing a bicubic trigonometric or bi-trigonometric Hermite B-spline surface patch interpolating the four edges of the resulting surface, we add four column vectors of control points ($P_{i,-2} = P_{i,-1} = P_{i,0}$ and $P_{i,n+3} = P_{i,n+2} = P_{i,n+1}$, $i = 0, 1, \cdots, n+1$) and four row vectors($P_{-2,j} = P_{-1,j} = P_{0,j}$ and $P_{n+3,j} = P_{n+2,j} = P_{n+1,j}$, $j = 0, 1, \cdots, n+1$).



(a) Control Points.



(b) $\lambda = -0.1$



(c) $\lambda = 1$

Figure 7: Bicubic B-spline surface patch with different parameters.

ACKNOWLEDGEMENTS

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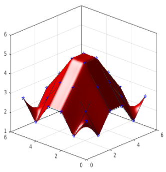
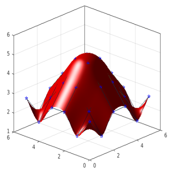
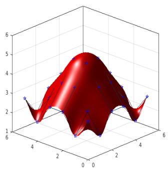


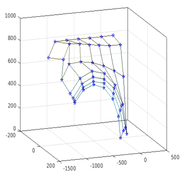
(a) $\lambda = 1 - \sqrt{2}$ and $\alpha = \frac{1}{4}$.



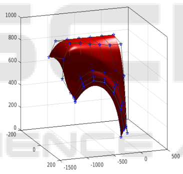


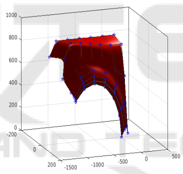
(b) $\lambda = 0.3$ and $\alpha = 0.4$. (c) $\lambda = 1$ and $\alpha = 0.45$.

Figure 8: Bicubic B-spline surface patch with different parameters.



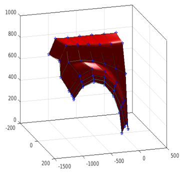
(a) Control Points.



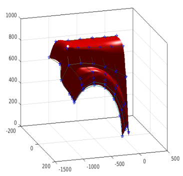


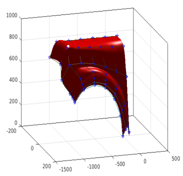
(b) $\lambda = 0$ (c) $\lambda = 1$

Figure 9: Bicubic B-spline surface patch with different parameters.



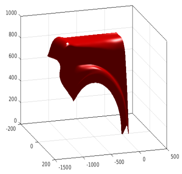
(a) $\lambda = 1 - \sqrt{2}$ and $\alpha = \frac{1}{4}$.



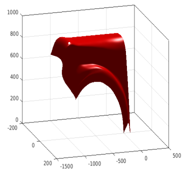


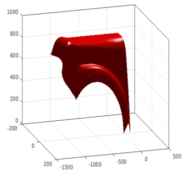
(b) $\lambda = 0.3$ and $\alpha = 0.4$. (c) $\lambda = 1$ and $\alpha = 0.45$.

Figure 10: Bicubic B-spline surface patch with different parameters.



(a) $\lambda = 1$.





(b) $\lambda = 0.3$ and $\alpha = 0.4$. (c) $\lambda = 1$ and $\alpha = 0.45$.

Figure 11: Bicubic B-spline surface patch with different parameters.

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