Fuzzy Contagion Cascades in Financial Networks

Giuseppe De Marco[1,2], Chiara Donnini[1], Federica Gioia[1] and Francesca Perla[1]

[1]*Department of Management and Quantitative Sciences, University of Naples Parthenope, Via Generale Parisi 13, Napoli 80132, Italy*

[2]*Center for Studies in Economics and Finance, University of Naples Federico II, Italy*

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Abstract: Previous literature shows that financial networks are sometimes described by fuzzy data. This paper extends classical models of financial contagion to the framework of fuzzy financial networks. The *degree of default* of a bank in the network consists in a (real valued) measure of the fuzzy default and it is computed as a fixed point for the dynamics of a modified *"fictitious default algorithm"*. Finally, the algorithm is implemented in MATLAB and tested numerically on a real data set.

1 INTRODUCTION

It is well known that the banking system is connected in a network by the mutual exposures that banks and other financial institutions assume towards each other and this kind of interbank exposures are recognized as a source of financial crisis known as contagion cascade. The literature on financial stability has increased significatively in last years (see for instance Glassermann and Young (2016) or Hurd (2016) for recent surveys); however, the issue of the lack of precise information about the overall interbank exposures in the system has not been exhaustively investigated. This is an important problem as banks are obliged to show their exposures within the balance sheet only few times per year. The present paper studies a financial network model under imprecise data in which interconnections are represented by fuzzy numbers and provide mathematical and computational tools in order to exploit the information arising from this model.

The paper by Eisenberg and Noe (2001) shows that obligations of all banks within the system are determined simultaneously by fixed point arguments. develops an algorithm[1] that converges to a *clearing payment vector* but, at the same time, gives information about the systemic risk in the systems.

On the other hand, Furfine (2003) is the first paper which studies the financial contagion arising from interbank exposures according to more realistic data that are based on daily observations along a two months period. Furfine find interbank exposures by looking at the transaction data in the Federal Reserve's large-value system (Fedwire). More precisely, he focuses only on *federal funds transactions* that are deduced from the Fedwire during February and March 1998[2]. Furfine (2003) found 719 commercial banks trading on the Fedwire and approximately 60000 federal fund transactions in the period taken into account. In Furfine's approach is that banks are classified into four groups according to the volume of funds traded. The exposure of a bank from one group in another bank from another group is expressed in terms of minimum, maximum and average value of the transactions between the two groups observed in the sample period. Furfine does not fully exploit the information arising from these data. Our interpretation of Furfine's data is instead that they can be read as triangular fuzzy numbers where the minimum and the maximum are obviously the inf and the sup of the support and the average is the maximum point for the membership function. This interpretation is the key motivation of our paper: on the one hand it shows that fuzzy data appear naturally in financial networks, on the other hand, we have already a detailed data set of fuzzy interbank exposures that can be used to run simulations.

[1]Known as the *fictitious default algorithm*.

[2]Furfine identifies federal funds transactions as follows: firstly, payments greater than \$1 million and ended in five zeros were identified as candidates. For each candidate payment, another payment between the same two banks in the opposite direction is searched the following day (plus interest). If such opposite payment is found, then the first payment is considered as federal fund exposure.

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In this paper, we follow the Eisenberg and Noe (2001) approach: we use fixed point arguments to show existence of clearing vectors and construct a suitable adjustment of the *fictitious default algorithm* to our fuzzy model. In our approach, the balance sheet of each bank is a triangular fuzzy number, called *fuzzy net worth*; then, we construct *index of fuzzy default* functions that assign to every fuzzy net worth a real (or crisp) *degree of fuzzy default*. For each degree of fuzzy default, clearing payment vectors are then constructed. We focus on two specific models: the optimistic δ model and the pessimistic σ model. They depend on the way the fuzzy net worth is *greater* than 0; namely, the optimistic model measures the set of all the alpha-cuts having not empty intersection with $\mathbb{R}_+$, while, the pessimistic model measures the set of all the alpha-cuts which are subsets of $\mathbb{R}_+$. The vector of degrees of default is characterized as a fixed point of the dynamics of a modified fictitious default algorithm. Finally, the algorithm is implemented in MATLAB using Furfine's (2003) data; simulations show that contagion spreads only within smaller banks as it was also shown in Furfine 2003.

2 FUZZY NUMBERS

In this section we recall some key notions and results from the theory of fuzzy numbers that are required in our model (see, for example, Buckley and Eslami (2002), Klir and Yuan (1995) and Zimmermann (2001) for extensive surveys and references).

Given a universal set X, a fuzzy subset A of X is a function which associates with each point in X a real number in the interval $[0,1]$. That function is called *membership function*.

A *fuzzy number n* is a particular fuzzy subset of $\mathbb{R}$, with membership function denoted by μ_n, such that

1. the core of n, i.e. the set $co(n) = \{x \in X \,|\, \mu_n(x) = 1\}$, is non-empty;
2. the α-cuts of n, i.e. the sets $\{x \in X \,|\, \mu_n(x) \geqslant \alpha\}$, are all closed, bounded, intervals, for every $\alpha \in]0,1]$[3];
3. the support of n, i.e. the set $supp(n) = \{x \in X \,|\, \mu_n(x) > 0\}$, is bounded[4].

[3]We remark that for each $\alpha \in]0,1]$ the α-cut of n is always $[\underline{n}(\alpha), \overline{n}(\alpha)]$. Moreover if $0 < \alpha_1 < \alpha_2 \leqslant 1$, then $n[\alpha_2] \subseteq n[\alpha_1]$.

[4]We remind that the *core* of n is $n[1]$, while the *support* of n is not $n[0]$, since the 0-cut is always the whole universal set. Moreover, since the support of n has to be bounded, there exists a positive real number l so that the support of n is a subset of $[-l,l]$.

A fuzzy number n is said to be *positive* if $\inf supp(n) > 0$, or, equivalently, if $supp(w_i) \subseteq \mathbb{R}_+$; n is said to be *negative* if $\sup supp(n) < 0$, or, equivalently, if $supp(w_i) \subseteq \mathbb{R}_-$. We can trivially observe that there are fuzzy numbers that are not positive neither negative. With abuse of notation we will indicate that n is positive (negative) with $n > 0$ ($n < 0$).

A *triangular fuzzy number n* is a continuous fuzzy number[5] such that the core is a singleton, i.e. $co(n) = \{\hat{n}\}$. Denote with $\underline{n} = \inf supp(n)$ and $\overline{n} = \sup supp(n)$, then the triangular fuzzy number n is denoted by $n = (\underline{n}, \hat{n}, \overline{n})$, while its membership function is defined as follows

$$\mu_n(x) = \begin{cases} \dfrac{x-\underline{n}}{\hat{n}-\underline{n}}, & \text{if } \underline{n} \leqslant x \leqslant \hat{n}; \\ \\ \dfrac{x-\overline{n}}{\hat{n}-\overline{n}}, & \text{if } \hat{n} < x \leqslant \overline{n}; \\ \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{N}$ the set of triangular fuzzy numbers. For the computation of the sum of triangular fuzzy numbers and the product of a triangular fuzzy number by a real number, we can use the following rule:

Given three triangular fuzzy numbers $n = (\underline{n}, \hat{n}, \overline{n})$, $m = (\underline{m}, \hat{m}, \overline{m})$, $l = (\underline{l}, \hat{l}, \overline{l})$ and a real number a,

$$i)\quad n+m-l = (\underline{m}+\underline{n}-\overline{l}, \hat{n}+\hat{m}-\hat{l}, \overline{n}+\overline{m}-\underline{l})$$
$$ii)\quad an = \begin{cases} (a\underline{n}, a\hat{n}, a\overline{n}), & \text{if } a > 0; \\ \\ (a\overline{n}, a\hat{n}, a\underline{n}), & \text{if } a < 0; \\ \\ 0, & \text{if } a = 0. \end{cases} \tag{1}$$

3 NETWORKS OF BANKS

Banks, Balance Sheets and Fuzzy Default

The market consists in a set of banks $I = \{1, 2, \ldots, n\}$. Each bank i is characterized by its balance sheet which, in turn, consists in *assets* and *liabilities*.

The bank's assets are:

i) Outside assets c_i: aggregate claims of bank i on nonfinancial entities;

ii) In-network assets p_{ki}, for each $k \neq i$. Each p_{ki} is the claim of bank i on bank k, that is, a payment obligation of bank k to bank i and is the aggregate exposure of bank i in the bank k.

The bank's liabilities include:

[5]A *continuous fuzzy number* is a fuzzy number having a continuous membership function.

i) Obligations b_i to nonfinancial entities;

ii) Obligations p_{ik}, for each $k \neq i$, to the bank k.

In the literature, the matrix $(p_{ik})_{i,k=1}^n$ is the adjacency matrix of an directed network, called *financial network*. Each node is a bank, and a directed edge runs from node i to node k if bank i has a payment obligation to node k. In this case we say that bank i is connected to bank k. All entities outside the network can be represented through a single node representing the "outside".

The difference between the bank i's assets and liabilities is the *net worth* w_i. Following the previous literature, we assume that all debt obligations have equal priority and the assets are distributed to creditors from each bank k in proportion η_k, where $\eta_k = 1$ if the bank k is able to honor all its debts *with certainty*, $\eta_k \in]0,1[$ if it is *possible* that k is not able to honor all its debts[6].

Therefore the *asset side* of node i's balance sheet is given by

$$c_i + \sum_{k \neq i} \eta_k p_{ki}$$

and the *liability side* by

$$b_i + \sum_{k \neq i} p_{ik}.$$

The node's *net worth* is

$$w_i = c_i + \sum_{k \neq i} \eta_k p_{ki} - b_i - \sum_{k \neq i} p_{ik}. \tag{2}$$

The previous formula of the net worth is standard in the literature on contagion (see Glasserman and Young (2016) or Hurd (2016)). Aim of this paper is to extend the previous in case of triangular fuzzy numbers. It is well known that in the crisp case, the default of a bank corresponds to a negative net worth. In the framework of the present paper, in which the net worth is a fuzzy number, the classical "binary" concept of default is inadequate. Therefore a suitable definition of *fuzzy default* must be given.

Definition 3.1. *We say that:*

i) A bank i defaults with certainty if its net worth $w_i < 0$, (i.e. $supp(w_i) \subseteq \mathbb{R}_-$).

ii) A bank i does not default if $w_i > 0$, (i.e. $supp(w_i) \subseteq \mathbb{R}_+$).

ii) A bank i incurs in a fuzzy default if $supp(w_i) \cap \mathbb{R}_- \neq \emptyset$[7].

[6] The term 'possible' refers to the situation of fuzzy default as it will be explained below.

[7] We remark that a certainty default is a particular case of a fuzzy default.

Degree of Default as a Fixed Point

In this subsection we construct a model which gives for every bank and every fuzzy net worth a reasonable vector of proportions η and a proper measure of default. The model will be assigned by a pair of functions (Λ, g) which specifies a supposed *degree* of default $\Lambda(\omega)$ given a fuzzy default ω (that is, Λ is defuzzified "degree of default"), and the proportions $\eta_k = g(\lambda_k)$ for every bank k and every degree of default λ_k. The pair (Λ, g) will be asked to satisfy specific properties.

Recall that for every triangular fuzzy number n, $\overline{n} = \sup supp(n)$, $\underline{n} = \inf supp(n)$ and $\hat{n}$ is the element of the core. Then,

Definition 3.2. *Let $\succsim_L$ be the binary relation on $\mathcal{N}$ defined by*

$$n \succsim_L m \iff \begin{cases} i) & \underline{n} \geqslant \underline{m}, \\ ii) & \overline{n} \geqslant \overline{m}, \\ iii) & \hat{n} \geqslant \hat{m}. \end{cases}$$

We say that n is L-related to m, if $n \succsim_L m$.

Moreover

Definition 3.3. *We say that a function $\Lambda : \mathcal{N} \to \mathbb{R}$ is L-decreasing if and only if*

$$n \succsim_L m \implies \Lambda(n) \geqslant \Lambda(m). \tag{3}$$

Then we introduce the model as follows

Definition 3.4. *A defuzzified measure of default is a pair of functions (Λ, g) where*

i) $\Lambda : \mathcal{N} \to [0,1]$ is a L-decreasing function such that

$$\Lambda(w) = \begin{cases} 0 & \text{if} \quad w > 0 \ (supp(w) \subseteq \mathbb{R}_+) \\ 1 & \text{if} \quad w < 0 \ (supp(w) \subseteq \mathbb{R}_-) \end{cases}$$

Λ is called index of fuzzy default and $\lambda_i = \Lambda(w_i)$ represents the degree of default of bank i when its net worth is w_i.

ii) $g : [0,1] \to [0,1]$ is a decreasing function such that $g(0) = 1$. The term $\eta_i = g(\lambda_i)$ gives the proportion of debts that bank i is "supposed" to distribute to the other banks if i incurs in a degree of default equal to λ_i.

Remark 3.5. *Every decreasing function g is suitable from a theoretical point of view even if, in examples and simulations, we will consider the simple functional form*

$$g(\lambda_i) = 1 - \lambda_i,$$

which represents a good approximation of the relation between the likelihood of default and expected rate of debt repayment.

Note also that the assumption of a decreasing relation between degree of default λ_i and the proportion η_i is natural as the greater is the likelihood of default the lower is the perception of solvability of i and, therefore, the lower is the expected rate of debt repayment.

Let (Λ, g) be a defuzzified measure of default. Denote with

$$H_i((\eta_k)_{k\neq i}) := c_i + \sum_{k\neq i} \eta_k p_{ki} - b_i - \sum_{k\neq i} p_{ik},$$

and with $F_i : [0,1]^{n-1} \to [0,1]$ the function defined by

$$F_i((\lambda_k)_{k\neq i}) = \Lambda\left(H_i\left((g(\lambda_k))_{k\neq i}\right)\right).$$

Let $F : [0,1]^n \to [0,1]^n$ be the function defined by

$$F(\lambda_1,...,\lambda_n) =$$
$$(F_1((\lambda_k)_{k\neq 1}),...,F_n((\lambda_k)_{k\neq n})) = (F_i((\lambda_k)_{k\neq i}))_{i=1,...,n}. \quad (4)$$

Then, it follows that

Proposition 3.6. *Let (Λ, g) be a defuzzified measure of default and $\lambda = (\lambda_1,...,\lambda_n) \in [0,1]^n$ where $\lambda_i = \Lambda(w_i)$ is the degree of default associated to a net worth w_i, for every bank $i = 1,\dots,n$. Then λ is a fixed point for F, i.e. $\lambda = F(\lambda)$.*

We have

Theorem 3.7. *Let (Λ, g) be a defuzzified measure of default, then the function F, defined as in (4), admits a fixed point.*

4 OPTIMISTIC AND PESSIMISTIC INDEXES OF FUZZY DEFAULT

This section focuses on two particular examples of indexes of fuzzy default which have an interesting interpretation and allow for simple computations.

Definition 4.1. *The function $\Lambda_\sigma : \mathcal{N} \to [0,1]$ defined by*

$$\Lambda_\sigma(w) = \begin{cases} \mu_w(0), & \text{if } \hat{w} \geqslant 0; \\ 1, & \text{if } \hat{w} < 0. \end{cases} \quad \forall w \in \mathcal{N} \quad (5)$$

is said to be pessimistic index of default.

and

Definition 4.2. *The function $\Lambda_\delta : \mathcal{N} \to [0,1]$ defined by*

$$\Lambda_\delta(w) = \begin{cases} 1-\mu_w(0), & \text{if } \hat{w} \leqslant 0; \\ 0, & \text{if } \hat{w} > 0; \end{cases} \quad \forall w \in \mathcal{N} \quad (6)$$

is said to be optimistic index of default.

Interpretation

1: If there is no default with certainty ($w > 0$), then $\Lambda_\delta(w) = \Lambda_\sigma(w) = 0$.

2: If there is default with certainty ($w < 0$), then $\Lambda_\delta(w) = \Lambda_\sigma(w) = 1$.

3: $\Lambda_\delta(w) \leqslant \Lambda_\sigma(w)\ \ \forall w \in \mathcal{N}$.

4: Suppose that there is fuzzy default with $\hat{w} > 0$. That is
$$\underline{w} < 0 < \hat{w} < \overline{w}$$
This fuzzy net worth represents the situation in which the values that are the most likely to occur are positive, but there is yet the possibility that negative values occur, even if with a small membership.
The optimistic index gives a degree $\Lambda_\delta(w) = 0$. The pessimistic index gives a degree $\Lambda_\sigma(w) = \mu_w(0)$ which is the measure of the range interval $[0, \mu_w(0)]$ of all the values α whose α-cuts include at least a negative value. Intuitively, the larger is the interval then more likely is the possibility of default.

5: Suppose that there is fuzzy default with $\hat{w} < 0$. That is
$$\underline{w} < \hat{w} < 0 < \overline{w}$$
This fuzzy net worth represents the situation in which the values that are the most likely to occur are negative but there is yet the possibility that positive values occur, even if with a small membership.
The pessimistic index gives a degree $\Lambda_\delta(w) = 1$. The optimistic index gives a degree $\Lambda_\delta(w) = 1 - \mu_w(0)$ which is the measure of the range interval $[\mu_w(0), 1]$ of all the values α whose α-cuts include all negative values. Intuitively, even in this case, the larger is the interval then more likely is the possibility of default.

6: The case $\underline{w} < \hat{w} = 0 < \overline{w}$ obviously gives $\Lambda_\delta(w) = 0$ and $\Lambda_\sigma(w) = 1$, which is the largest possible differences between the two degrees. This sounds reasonable as $\hat{w} = 0$ is the case in which uncertainty is maximal, since there are no reasons to believe that negative values occur more likely than positive ones and viceversa.

5 FUZZY CONTAGION

In this section, we propose a model of default cascade in the our framework in which net worths are triangular fuzzy numbers. In particular, we extend the classical *"fictitious default algorithm"* introduced in Eisenberg and Noe (2001) (see also Glasserman and Young (2016)) to our fuzzy framework. The dynamic process is constructed in general for arbitrary defuzzified measures of fuzzy contagion, but we will look also at the particular cases of the Λ_δ and Λ_σ functions defined in the previous section.

The Contagion Dynamics

The exogenous shock is parametrized as a vector

$$x = (x_1, \dots, x_n)$$

where each x_i is a triangular fuzzy number which represents the exogenous shock which affects the (ante-shock) capital of bank i which, in turn, is characterized by the difference $b_i - c_i$[8]. In particular, we study the fuzzy default cascade step by step, computing in each step the net worth and the associated degree of default and show the convergence of the dynamics to a fixed point.

In particular the cascade is constructed as follows:

- at step $h = 1$ the exogenous shock x occurs. The net worth of each bank i is given by:

  $$w_i^1 = H_i((\eta_k^0)_{k\neq i}) = c_i + \sum_{k\neq i} \eta_k^0 p_{ki} - b_i - \sum_{k\neq i} p_{ik} - x_i,$$

  where each $\eta_k^0 = 1$ meaning that every bank is solvable before the exogenous shock.

  For each i, we then compute

  $$\lambda_i^1 = \Lambda(w_i^1), \quad \eta_i^1 = g(\lambda_i^1).$$

- at each step h, given the the vectors

  $$\begin{cases} w^{h-1} = (w_1^{h-1}, w_2^{h-1}, \dots, w_n^{h-1}) \\ \lambda^{h-1} = (\lambda_1^{h-1}, \lambda_2^{h-1}, \dots, \lambda_n^{h-1}) \\ \eta^{h-1} = (\eta_1^{h-1}, \eta_2^{h-1}, \dots, \eta_n^{h-1}), \end{cases}$$

  we compute, for each i,

  $$\begin{cases} w_i^h & = H_i((g(\lambda_k^{h-1}))_{k\neq i}) \\ & = c_i + \sum_{k\neq i} g(\lambda_k^{h-1}) p_{ki} - b_i - \sum_{k\neq i} p_{ik} - x_i \\ \lambda_i^h & = \Lambda(w_i^h) \\ \eta_i^h & = g(\lambda_i^h), \end{cases}$$

- therefore we get a sequence $(\lambda^h)_{h\in\mathbb{N}} \subset [0,1]^n$ as follows: By construction,

  $$\lambda_i^h = \Lambda\left(H_i((g(\lambda_k^{h-1}))_{k\neq i})\right) =$$

  $$F_i\left((\lambda_k^{h-1})_{k\neq i}\right) \quad \forall i = 1, \dots, n; \forall h \in \mathbb{N};$$

  being $F = (F_1, \dots, F_n)$, it follows that

  $$\lambda^h = F\left(\lambda^{h-1}\right) \quad \forall h \in \mathbb{N}.$$

It immediately follows that the stationary points for this sequence are fixed points for the function F, called degree of fuzzy default. We will show below the convergence of the sequences in the optimistic and in the pessimistic models.

Remark 5.1 (Simulations)**.** *The algorithm, proposed in the present paper, has been implemented in* MATLAB *and tested numerically on a real financial data set in order to analyze the contagion dynamics for the case of* fuzzy *input data. The analysis of the result of our simulation confirms Furfine's prediction that only small banks may be affected by contagion.*

[8] Indeed the exogenous shock could also have been implicitly included in the capital $b_i - c_i$ without any exogenous parameter x. Therefore it could be possible to study the financial contagion for every choice of capital $b_i - c_i$.

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