

e-Learning Platform Ranking Method using a Symbolic Approach based on Preference Relations

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Abstract: e-Learning platforms are of a great help in teaching and learning fields given their ability to improve training activity quality. Subsequently, several e-Learning systems have been developed in many domains. The diversity of such platforms in a single field makes it arduous to select the optimal platform in terms of tools and services that meet users' requirements. Therefore, we propose in this paper a ranking approach of e-Learning platforms relying on symbolic values, borrowed from the Qualitative Weight and Sum method (QWS) (Stufflebeam, 1994), preference relations and aggregating operators providing a total order among the considered e-Learning platforms.

1 INTRODUCTION AND MOTIVATION

The past decade has seen tremendous changes in educational and industrial training methods along with the increasing of the number of users having diverse needs and objectives. Indeed, there are a huge number of free and commercial e-Learning systems which have been developed in different areas such as education (Venkataraman and Sivakumar, 2015), language learning (Bañados, 2013), business training (Colace et al., 2006; Ubell, 2000), medicine (Schneider et al., 2015; Hannan, 2013) and public administrations (Stoffregena et al., 2015), etc. which provide on-line and remote training making user learning tasks more flexible and easier. The multitude of e-learning platforms developed for a single domain (such as in language learning, for instance, we can distinguish tens of e-learning applications and on-line platforms like *babel*, *busuu*, *duolingo*, *ef*, *tell me more*, *Pimsleur*, etc.) makes it difficult to pick the more suitable one according to one's needs and objectives.

The choice of a suitable system in compliance with user's needs and goals based on some criteria is important. Some criteria are mandatory to choose platform but they are insufficient, such as the compatibility of the e-Learning system on hand to certain norms and standards like SCORM¹,

¹SCORM: Sharable Content Object Reference Model, <http://scorm.com>

QTI², IMS³, etc. These standards ensure a structured learning object creation and e-Learning quality through properties, such as adaptability, sustainability, interoperability and reusability. We refer the reader to (García and Jorge, 2006) for an e-Learning platform evaluation based on the SCORM specification.

Besides, many other evaluation approaches have been proposed such as (Britain and Liber, 2004), in which the framework considers two models. The former addresses the different ways to produce learning processes in an e-Learning system, which has been reused in (Laurillard, 2013), and the latter characterizes the different evaluation criteria of learning models as introduced in (Liber et al., 2000).

Qualitative methods have also been considered for e-Learning systems evaluation; the most commonly used one is Qualitative Weight and Sum, denoted by QWS (Stufflebeam, 1994). It relies on a list of weighted criteria (Graf and List, 2005; Hamtini and Fakhouri, 2012) for the evaluation of e-Learning systems. In practice, it is based on qualitative weight symbols expressing six levels of importance, namely: *E* for *essential*, *** for *extremely valuable*, *#* for *very valuable*, *+* for *valuable*, *|* for *marginally valuable*, and *0* for *not valuable*. Hence, e-Learning system's

²QTI: Question and Test Interoperability, <http://www.ims-global.org>

³IMS: Instructional Management Systems, <http://www.ims-global.org>

performance is measured by symbolic weights attached to some criteria as described in (Graf and List, 2005), such that low-weighted criteria cannot overpower high-weighted ones. For instance, if a criterion weighted #, the platform can only be judged # or lesser (+, | or 0) but not * or higher. To obtain a global evaluation for a platform, QWS approach aggregates the symbols attached to criteria through a simple counting, which is finally used to rank the considered e-learning systems. Because of the naive aggregation function used by the approach, the result may be counterintuitive and not clear to explain and justify. For example, let us suppose three e-Learning systems, denoted by e_1, e_2, e_3 respectively, for which the aggregation function delivers the results as summarized in Table 1. It is easy to conclude that e_1 is

Table 1: Example of e-Learning system aggregation results.

	E	*	#	+		0
e_1	-	3	4	-	2	-
e_2	-	2	4	-	2	-
e_3	-	2	8	1	2	-

better than e_2 , since e_1 is better than e_2 on symbol * and both tied the score for the other symbols. But, it is not that easy to say whether e_1 is better than e_3 or not, because even though e_1 performs well on symbol *, e_3 is much more better than e_1 on symbols # and +. In the latter case, further analysis has to be conducted to conclude. As some e-learning systems are not comparable, then the approach delivers a pre-order over the evaluated platforms.

To deal with this issue, one can consider the Analytic Hierarchy Process (AHP) method (Hamtni and Fakhouri, 2012). AHP is used to deal with complex decision-making processes. It translates the symbols defined in QWS into values as detailed in Table 2, borrowed from (Stufflebeam, 1994). Thus, AHP captures both subjective and objective values, checks their consistency and reduces bias decision making in testing and evaluating e-Learning systems (Maruthur et al., 2015). The criteria are gathered up by category and sub-category. The results of the feature’s category or subcategory evaluation computed by the weight calculation functions are percentages of the form of a real number as described in (Hamtni and Fakhouri, 2012). For example, let us say that the percentage returned for the feature *Chat* is 14%. Then, according to Table 2, the judgment of this result is between “marginally valuable” and “valuable”, but which is it? The percentages returned can be difficult to interpret for comparing e-learning platforms when several attributes have to be dealt with.

As these methods return numerical values or av-

Table 2: QWS symbols translated into AHP weights.

	QWS	Weight in AHP
Essential	E	5
Extremely valuable	*	4
Very valuable	#	3
Valuable	+	2
Marginally valuable		1
Not valuable	0	0

erages (Graf and List, 2005), which can be less expressive and non-intuitive enough from a user standpoint for system quality assessment and ranking, then we propose in this paper a hybrid approach for system assessment and ranking combining QWS values, symbolic preference relations and formal comparison operators, which have been proved to be total orders allowing the distinction of optimal e-Learning platforms from the user standpoint.

The remainder of the paper is structured as follows. Section 2 details our symbolic-based approach for e-Learning systems evaluation. Section 3 presents an illustrative example of our approach to evaluate and to rank a set of open-source e-Learning systems. Finally, section 4 concludes the paper and introduces some future work.

2 HYBRID E-LEARNING SYSTEM EVALUATION APPROACH

In this section, we detail our approach for e-Learning platform evaluation and ranking relying on symbols borrowed from QWS method and qualitative preference relation and comparison operators. In Subsection 2.1, we introduce our evaluation approach and in Subsection 2.2, we show the use of our approach for e-learning platform ranking.

2.1 Symbolic Approach for e-Learning Platforms Evaluation

We define the evaluation symbols as follows.

Definition 1 (Evaluation Symbols). The evaluation symbols as defined in QWS approach are:

$E = \text{essential}$, $* = \text{extremely valuable}$, $\# = \text{very valuable}$, $+ = \text{valuable}$, $| = \text{marginally valuable}$ and $0 = \text{not valuable}$. We denote by $S = \{E, *, \#, +, |, 0\}$ an ordered set of evaluation symbols.

We define a preference relations *more preferred than or equal to*, denoted by \succeq , and *less preferred*

than or equal to, denoted by \preceq , over the evaluation symbol set \mathcal{S} as follows.

Definition 2 (Preference relations \succeq and \preceq). Let $\mathcal{S} = \{E, *, \#, +, |, 0\}$ be an ordered set of evaluation symbols such that:

- Position 1 is for symbol E , denoted by $pos_{\mathcal{S}}(E)$
- Position 2 is for symbol $*$, denoted by $pos_{\mathcal{S}}(*)$
- Position 3 is for symbol $\#$, denoted by $pos_{\mathcal{S}}(\#)$
- Position 4 is for symbol $+$, denoted by $pos_{\mathcal{S}}(+)$
- Position 5 is for symbol $|$, denoted by $pos_{\mathcal{S}}(|)$
- Position 6 is for symbol 0 , denoted by $pos_{\mathcal{S}}(0)$

We define the preference relation *more preferred than or equal to* \succeq over \mathcal{S} as follows.

$$\forall(a, b) \in \mathcal{S}^2 : a \succeq b \text{ iff } pos_{\mathcal{S}}(a) \leq pos_{\mathcal{S}}(b) \quad (1)$$

The preference relation *less preferred than or equal to*, denoted \preceq , is defined as follows.

$$\forall(a, b) \in \mathcal{S}^2 : a \preceq b \text{ iff } pos_{\mathcal{S}}(a) \geq pos_{\mathcal{S}}(b) \quad (2)$$

We can easily prove that the preference relation \succeq is a total order.

Property 1. (Total order properties). The preference relations \succeq and \preceq are a total order.

Proof. The proof of property 1 is detailed in Appendix A. \square

Based on the above defined preference relations, we define two comparison operators named *prefMin* and *prefMax*, so that it will be possible to compare systems on each criterion describing them. These operators will serve as means to aggregate the evaluations obtained for system criteria.

Definition 3. (*prefMax* and *prefMin* comparison operators). *prefMax* and *prefMin* operators are defined by formulas (3) and (4) respectively.

The function *prefMax* is defined by the following formula (3).

$$\begin{aligned} \mathcal{S} \times \mathcal{S} &\rightarrow \mathcal{S} \\ (a, b) &\mapsto \max(a, b) = \begin{cases} a & \text{if } (a \succeq b) \\ b & \text{otherwise.} \end{cases} \end{aligned} \quad (3)$$

The function *prefMin* is defined by the following formula (4).

$$\begin{aligned} \mathcal{S} \times \mathcal{S} &\rightarrow \mathcal{S} \\ (a, b) &\mapsto \min(a, b) = \begin{cases} a & \text{if } (a \preceq b) \\ b & \text{otherwise.} \end{cases} \end{aligned} \quad (4)$$

When we apply the comparison operators *prefMax* and *prefMin* over our symbolic set \mathcal{S} , we obtain Table 3.

Property 2. (*prefMax* properties). *prefMax* is associative, commutative, idempotent, it has E as absorbent element and 0 as neutral element.

Proof. Proofs of *prefMax* properties are detailed in Appendix B. \square

Property 3. (*prefMin* properties). *prefMin* is associative, commutative, idempotent, it has 0 as absorbent element and E as neutral element.

Proof. Proofs of *prefMin* properties are detailed in Appendix C. \square

2.2 Using our Comparison Operators to Rank e-Learning Systems

The evaluation of e-Learning platforms is based on categories, each of which defines some criteria as defined in (Atthirawong and MacCarthy, 2002), for example the category *Communication tools*, and their criterion such as *Chat*. Categories and their criteria are summarized in Table 4. The five categories considered in platform evaluation are the following:

- Communication tools
- Software and installation
- Administrative tools and security
- Hardware presentation tools
- Management features

To evaluate each category, we use the comparison operators *prefMax* and *prefMin*. But to evaluate a considered e-Learning system, we need the evaluation of the five categories. For that purpose, we define two aggregation operators, called *prefMinMax* and *prefMaxMin*, which are based on our comparison operators.

Definition 4. (*prefMinMax*). Let \mathcal{A} be a matrix of n lines and m columns of evaluation symbols of \mathcal{S} . We define the minimum guaranteed satisfaction value as follows.

We denote a matrix from \mathcal{A} as:

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \text{ and } a_{ij} \in \mathcal{S}$$

We define *prefMinMax* of A as:

$$\begin{aligned} \mathcal{S}^{m \times n} &\rightarrow \mathcal{S} \\ A &\mapsto \text{prefMinMax}(A) = \\ &\text{prefMin}_{1 \leq i \leq m}(\text{prefMax}_{1 \leq j \leq n}(a_{ij})) \end{aligned} \quad (5)$$

Table 3: The operators *prefMax* and *prefMin* table.

<i>prefMax</i>	<i>E</i>	*	#	+		0
<i>E</i>	<i>E</i>	<i>E</i>	<i>E</i>	<i>E</i>	<i>E</i>	<i>E</i>
*	<i>E</i>	*	*	*	*	*
#	<i>E</i>	*	#	#	#	#
+	<i>E</i>	*	#	+	+	+
	<i>E</i>	*	#	+		
0	<i>E</i>	*	#	+		0

<i>prefMin</i>	<i>E</i>	*	#	+		0
<i>E</i>	<i>E</i>	*	#	+		0
*	*	*	#	+		0
#	#	#	#	+		0
+	+	+	+	+		0
						0
0	0	0	0	0	0	0

Table 4: Overview of the evaluation hierarchy categories and their criteria.

Category	Communication tools	Software & Installation	Administrative tools and Security	Hardware Presentation tools	Management features
Criterion	Chat Forum Mail Video conference Calendar	Downloading Installation Assistance Documentation	Courses administration Tracking progress Online registration Learning path creation Report Learning path organisation Test evaluation Security	Announcements Learning Objects Exercises Content import	Multi course management Multi user management Evaluation management User Group

Definition 5. (*prefMaxMin*). We define the maximum possible satisfaction value of $S^{m \times n}$ as *prefMaxMin*:

$$\begin{aligned}
 S^{m \times n} &\rightarrow S \\
 A &\mapsto \text{prefMaxMin}(A) = \\
 &\text{prefMax}_{1 \leq i \leq m}(\text{prefMin}_{1 \leq j \leq n}(a_{ij}))
 \end{aligned}
 \tag{6}$$

The *prefMinMax* operator computes the least optimistic value amongst the criteria, whereas *prefMaxMin* operator computes the greatest pessimistic value amongst the criteria.

3 ILLUSTRATIVE EXAMPLE

We apply our e-Learning systems evaluation approach to a set of nine open-source e-Learning enumerated below, and which have been tested and compared their (Lebrun et al., 2008),(Reiter et al., 2006) (Dogbe Semanou et al., 2007) (Lafordade and Oubahssi, 2014).

1. Claroline: version 1.9.2, <http://www.claroline.net>
2. Dokeos: version 2.1.1, <http://www.dokeos.com/fr>
3. eFront: version 3.6.11, <http://www.efrontlearning.net>
4. ILIAS: version 4.1.3, <http://www.ilias.de>

5. Open ELMS: version 7, <http://www.openelms.org>
6. Ganesha: version 4.5, <http://www.ganesha.fr>
7. Olat: version 7.2.1, <http://www.olat.org>
8. AnaXagora: version 3.5, <http://www.anaxagora.tudor.lu>
9. Sakai: version 10.4, <https://sakaiproject.org>

Table 5: *prefMin* and *prefMax* results for Communication Tools category.

Category	Communication tools					<i>prefMin</i>	<i>prefMax</i>
	chat	Forum	Mail	Conference Video	Calendar		
Criterion							
Claroline	#	#	+	#	+	+	#
Dokeos	*	+	+	*	*	+	*
eFront		#	#	+	+		#
ILIAS	#	+	+	0	+	0	#
Open ELMS	0	0	*	0	0	0	*
Ganesha	#	#	+	0	0	0	#
Olat	*	*	*	0	*	0	*
AnaXagora	#	#	#	0	+	0	#
Sakai	*	#	*	*	#	#	*

Table 6: Results of *prefMaxMin* computation over the set of e-learning platform.

Category	Communication tools	Software & Installation	Administrative tools and Security	Hardware presentation tools	Management features	<i>prefMaxMin</i>
Platform	<i>prefMin</i>	<i>prefMin</i>	<i>prefMin</i>	<i>prefMin</i>	<i>prefMin</i>	
Claroline	+	+	+	#	+	#
Dokeos	+			#	+	#
eFront		#				#
ILIAS	0			0		
Open ELMS	0			0	0	
Ganesha	0		+	0	#	#
Olat	0			0		
AnaXagora	0			0		
Sakai	#			0		#

Table 7: Results of *prefMinMax* computation over the set of e-learning platform.

Category	Communication tools	Software & Installation	Administrative tools and Security	Hardware presentation tools	Management features	<i>prefMinMax</i>
Platforms	<i>prefMax</i>	<i>prefMax</i>	<i>prefMax</i>	<i>prefMax</i>	<i>prefMax</i>	
Claroline	#	*	*	*	#	#
Dokeos	*	#	E	*	E	#
eFront	#	*	*	+	+	+
ILIAS	#	+	#	#	+	+
Open ELMS	*	+	*	*	#	+
Ganesha	#	+	E	#	#	+
Olat	*	#	*	*	*	#
AnaXagora	#	+	*	#	*	+
Sakai	*	*	E	*	E	*

Each criterion takes a symbolic value from the set *S* based on users opinions community. To obtain the evaluation of each criterion, we have carried out sur-

veys in our university involving under-graduated students (small group of 10 students), who have tested each e-learning platform during a training session (2

Table 8: E-Learning platform's features evaluation.

Criterion	Category	Communication tools	Software & Installation	Administrative tools and Security	Hardware presentation tools	Management features
Claroline		chat	Downloading	Courses administration	Announcements	Multi course management
Dokeos		Forum	Installation	Tracking progress	Learning Objects	Multi user management
eFront		Mail	Assistance	Online registration	Exercises	Evaluation management
ILIAS		Video conference	Documentation	Learning path creation	Content import	User group
Open ELMS		Calendar		Report		
Ganesha				Learning path organisation		
Olat				Test evaluation		
AnaXagora				Security		
Sakai						

hours). We are aware that the process is subjective and a different panel of students or users can express different opinions about the e-Learning platforms. We recall that this data collection aims at illustrating the use of our approach. The values obtained for each criterion in its category are summarized in Table 8. The application of our approach on the set of considered systems is performed as follows.

1. for each category in Table 4 we calculate values of $prefMin$ and $prefMax$ for all functionalities based on Definition 3. In Table 5, we display the results obtained by applying our approach on the category “Communication Tools” for our considered set of e-learning platforms.
2. for all categories in Table 8 we calculate values of $prefMaxMin$ and $prefMinMax$. Results of both calculus are displayed in Table 6 and 7 respectively.

According to Table 6, we obtain the following ranking over the set of e-learning system considered.

1. Claroline, Dokeos, eFront, Ganesha and Sakai.
2. Ilias, Open ELMS, Olat and AnaXagora.

According to Table 7, we obtain the following ranking over the set of e-learning system considered.

1. Sakai
2. Claroline, Dokeos and Olat
3. eFront, ILIAS, Open ELMS, Ganesha and AnaXagora

Finally, users can make a choice based on either $prefMaxMin$ or $prefMinMax$ operators or can combine the result returned by both. For instance, in our illustrative example, Claroline, Dokeos, eFront, Ganesha and Sakai are all optimal platforms according to $prefMaxMin$ operator, whereas Sakai is the optimal one according to $prefMinMax$ operator. But, we can notice that Sakai performs better since it is optimal according to both operators.

4 CONCLUSION AND FUTURE WORK

In this paper, we have presented an e-Learning systems evaluation approach based on a symbolic set of value, a total order preference relation and comparison operators. To describe e-Learning system, we have used categories, each of which defines some criterion of well-known properties of these systems. We apply our approach on a set of open source e-Learning systems for which you have gathered through small

surveys their evaluation on the considered criteria. The proposed approach assesses the quality of an e-Learning system amongst a set of e-Learning platforms by considering a maximum possible satisfaction and/or a minimum guaranteed satisfaction. Once this value is obtained, it becomes easy to rank the set of e-learning systems considered from the most to the least satisfactory, and to deliver to the user the one or several optimal systems.

Our approach brings a solution to the problem of choosing a system according to well-defined criteria. It is still to perform a larger survey to obtain values as accurate as possible for the criteria. It is also worthy to consider user profiles when performing surveys in such a way that we obtain different values for different profiles. A profile can be defined over a population of users based on their interests and training objectives.

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Appendix

A Proof of Property 1 Total Order

We only prove hereinafter the property for the preference relation \succeq . The proof of the property

for the preference relation \preceq is similar to the one of \succeq .

Proof 1. (\succeq is total order) The preference relation \succeq is a total order iff:

1. \succeq is reflexive
 2. \succeq is antisymmetric
 3. \succeq is transitive
1. Relation \succeq is reflexive iff $\forall a \in S : a \succeq a$. Therefore, $a \succeq a$ iff $pos_S(a) \leq pos_S(a)$ which is verified for the comparison operator \leq since \leq is reflexive. Then \succeq is reflexive.
2. Relation \succeq is antisymmetric iff $\forall a, b \in S : a = b$. Then:
 $a \succeq b \wedge b \succeq a$ iff $pos_S(a) \leq pos_S(b) \wedge pos_S(b) \leq pos_S(a)$ which is verified for \leq since \leq is antisymmetric.
 Then \succeq is antisymmetric.
3. \succeq is transitive iff $\forall a, b, c \in S : a \succeq b \wedge b \succeq c \Rightarrow a \succeq c$. As $a \succeq b \wedge b \succeq c$ then $pos_S(a) \leq pos_S(b) \wedge pos_S(b) \leq pos_S(c)$. Therefore, $pos_S(a) \leq pos_S(c)$ since \leq is transitive. That means that $a \succeq c$ and \succeq is transitive.

B Proof of *prefMax* Properties

Proof 2. (*prefMax* properties).

1. *prefMax* is associative on S :
 $\forall a, b, c \in S$, then: $prefMax(prefMax(a, b), c) = prefMax(a, prefMax(b, c))$. We denote by I the left term $prefMax(prefMax(a, b), c)$ and by II the right term $prefMax(a, prefMax(b, c))$.
 Table 9 shows results of evaluation of the left and the right terms, which are identical. Therefore *prefMax* is associative.

Table 9: The formula results.

	I	II
$a \succeq b \wedge a \succeq c$	a	$\Rightarrow a \succeq prefMax(b, c) \Rightarrow II = a$
$a \succeq b \wedge c \succeq a$	c	$\Rightarrow c \succeq b$ (transivity) $prefMax(b, c) = c \Rightarrow II = c$ since $c \succeq a$
$b \succeq a \wedge b \succeq c$	b	$prefMax(b, c) = b \Rightarrow II = b$ since $b \succeq a$
$b \succeq a \wedge c \succeq b$	c	$prefMax(b, c) = c \wedge c \succeq a$ (transivity) $\Rightarrow II = c$

2. *prefMax* is commutative iff $\forall a, b \in S : prefMax(a, b) = prefMax(b, a)$.
 From table 3, the *prefMax* matrix is symmetric so *prefMax* is commutative.

- 3. $prefMax$ is idempotent iff $\forall a \in S : prefMax(a,a) = a$.
From the main diagonal of table 3, we conclude that $prefMax$ is idempotent.
- 4. $prefMax$ has 0 as neutral element iff $\forall a \in S : prefMax(a,0) = a$.
Table 3 shows that $prefMax$ has 0 as neutral element.
- 5. $prefMax$ has E as absorbent element iff $\forall a \in S : prefMax(a,E) = E$.
Table 3 also shows that $prefMax$ has E as absorbent element.
- 5. $prefMin$ has 0 as absorbent element iff $\forall a \in S : prefMin(a,E) = E$.
Table 3 also shows that $prefMin$ has 0 as absorbent element.

C Proof of $prefMin$ Properties

Proof 3. ($prefMin$ properties)

- 1. $prefMin$ is associative on S :
 $\forall a,b,c \in S: prefMin(prefMin(a,b),c) = prefMin(a,prefMin(b,c))$. We denote by I the left term $PrefMin(PrefMin(a,b),c)$ and by II the right term $PrefMin(a,PrefMin(b,c))$.
Table 10 shows results of evaluation of the left and the right terms, which are identical. Therefore $prefMin$ is associative.

Table 10: The formula results.

	I	II
$a \succeq b \wedge b \succeq c$	c	$\Rightarrow a \succeq prefMin(b,c) \Rightarrow II = c$ since $a \succeq c$
$a \succeq b \wedge c \succeq b$	b	$\Rightarrow a \succeq prefMin(b,c) \Rightarrow II = b$ since $a \succeq b$
$b \succeq a \wedge a \succeq c$	c	$\Rightarrow b \succeq c$ (transitivity) $prefMin(b,c) = c \Rightarrow II = c$ since $a \succeq c$
$b \succeq a \wedge c \succeq a$	a	a is the smallest symbol between a, b and c , so $II = a$

- 2. $prefMin$ is commutative iff $\forall a,b \in S : prefMin(a,b) = prefMin(b,a)$.
From table 3, the $prefMax$ matrix is symmetric so $prefMin$ is commutative.
- 3. $prefMin$ is idempotent iff $\forall a \in S : prefMin(a,a) = a$.
From the main diagonal of table 3, we conclude that $prefMin$ is idempotent.
- 4. $prefMin$ has E as neutral element iff $\forall a \in S : prefMin(a,0) = a$.
Table 3 shows that $prefMin$ has E as neutral element.