# Fuzzy Semi-Quantales, (L,M) Quasi-Fuzzy Topological Spaces and Their Duality

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Abstract: The present paper introduces $M$-fuzzy semi-quantales, fuzzifying semi-quantales, and $(L,M)$-quasi-fuzzy topological spaces, providing a common framework for $(L,M)$-fuzzy topological spaces of Kubiak and Šostak, $L$-quasi-fuzzy topological spaces of Rodabaugh and $L$-fuzzy topological spaces of Höhle and Šostak. In this paper, we set up a dual adjunction between the category of $(L,M)$-quasi-fuzzy topological spaces and the category of $M$-fuzzy semi-quantales, and then show that this adjunction includes a dual equivalence between the category of $(L,M)$-sober $(L,M)$-quasi-fuzzy topological spaces and the category of $(L,M)$-spatial $M$-fuzzy semi-quantales.

## 1 INTRODUCTION

Topological duals of ordered algebraic structures have been a fundamental issue in mathematics (Clark and Davey, 1998; Erné, 2004 ; Jhonstone, 1986; Lawson, 1979) since the seminal paper of M. H. Stone (Stone, 1936) on the representations of Boolean algebras. Most known fuzzy topological duals of (fuzzy) ordered algebraic structures in fuzzy mathematics (Demirci, 2014; Höhle, 2001; Solovyov, 2008; Yao, 2012) have been inspired by the famous Papert-Papert-Isbell adjunction (Isbell, 1972; Jhonstone, 1986; Papert and Papert, 1957/1958) between the category **Top** of topological spaces and the opposite $\mathbf{Frm}^{op}$ of the category **Frm** of frames. $(L,M)$-fuzzy topological spaces of Kubiak and Sostak (Kubiak and Šostak, 2009), $L$-quasi-fuzzy topological spaces of Rodabaugh (Rodabaugh, 2007), $L$-fuzzy topological spaces of Höhle and Šostak (Höhle and Šostak, 1999) are three important approaches to the fuzzy topological spaces in which fuzzy topologies are assumed to be fuzzy sets themselves. In order to provide a common framework for these approaches, we propose $(L,M)$-quasi-fuzzy topological space, and formulate their category $(L,M)$-**QFTop** on the basis of fixed semi-quantales $L$ and $M$. Our aim in this paper is to find out suitable (possibly fuzzy) ordered algebraic structures, whose topological counterparts are $(L,M)$-quasi-fuzzy topological spaces, and is to set up an analog of the Papert-Papert-Isbell adjunction for $(L,M)$-quasi-fuzzy topological spaces and such ordered algebraic structures. More clearly, we introduce $M$-fuzzy semi-quantales as asked ordered algebraic structures, and build the category $M$-**FSQuant** of them in the next section. Section 3 is devoted to $(L,M)$-quasi-fuzzy topological spaces and their categories. We reserve the fourth section for the main contributions of this paper: the adjunction between $(L,M)$-**QFTop** and the dual of $M$-**FSQuant**, and the refinement of this adjunction to a dual equivalence between the full subcategory of $(L,M)$-**QFTop** of all $(L,M)$-sober objects and the full subcategory of $M$-**FSQuant** of all $(L,M)$-spatial objects. This dual equivalence, providing the representation of $M$-fuzzy semi-quantales by means of $(L,M)$-quasi-fuzzy topological spaces, can also be thought of as an analog of the famous Stone duality (Jhonstone, 1986) between Boolean algebras and compact, Hausdorff and zero-dimensional topological spaces (alias Stone spaces), where we replace Boolean algebras and Stone spaces by $(L,M)$-spatial $M$-fuzzy semi-quantales and $(L,M)$-sober $(L,M)$-quasi-fuzzy topological spaces, respectively. In Section 5, we show that the relationships in Section 4 can also be enlarged to the categories of strong $M$-fuzzy semi-quantales and of unital $M$-fuzzy semi-quantales.



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## 2 FUZZY SEMI-QUANTALES

Fuzzy semi-quantale is a fuzzification of the concept of semi-quantale. A semi-quantale (an s-quantale for short) $(L,\leq,\otimes)$ is defined to be a complete lattice $(L,\leq)$ with a binary operation $\otimes : L\times L\rightarrow L$, called a tensor product (Rodabaugh, 2007). As convention, we denote the join, meet, top and bottom elements in the complete lattice $(L,\leq)$ by $\bigvee$, $\bigwedge$, $\top_L$ and $\bot_L$, respectively. S-quantales include various classes of ordered algebraic structures (e.g., complete residuated lattices, unit interval $[0,1]$ equipped with uninorms (t-norms or t-conorms in particular), quantales, frames, semi-frames) playing a major role in fuzzy logics and fuzzy set theory (Bělohlávek, 2002; Hájek, 1998; Novák, Perfilieva and Močkoř, 1999). Now we give only some of their definitions that will be needed in the following text.

**Definition 1.** *(i) An s-quantale $(L,\leq,\otimes)$ is called a unital s-quantale, abbreviated as us-quantale if $\otimes$ has an identity element $e\in L$ called the unit (Rodabaugh, 2007).*

*(ii) A us-quantale $(L,\leq,\otimes)$ with $e=\top_L$ is called a strictly two-sided s-quantale, abbreviated as st-s-quantale (Rodabaugh, 2007).*

*(iii) An s-quantale $(L,\leq,\otimes)$ is called a complete groupoid if $\otimes$ is isotone in both variables (Höhle, 2001).*

*(iv) A complete groupoid $(L,\leq,\otimes)$ is called a complete quasi-monoidal lattice, abbreviated as cqm-lattice if $x\leq x\otimes\top_L$ and $x\leq\top_L\otimes x$ hold for all $x\in L$ (Höhle and Šostak, 1999).*

*(v) An s-quantale $(L,\leq,\otimes)$ is called a quantale if $\otimes$ is associative and distributive over arbitrary $\bigvee$ (Rosenthal, 1990).*

All s-quantales constitute a category **SQuant** with morphisms (the so-called s-quantale morphisms) all functions preserving $\otimes$ and arbitrary $\bigvee$ (Rodabaugh, 2007). An s-quantale morphism is said to be strong if it preserves the top element (Demirci, 2010). S-quantales together with strong s-quantale morphisms form a non-full subcategory **SSQuant** of **SQuant** (Demirci, 2010). **USQuant** is another one of the non-full subcategories of **SQuant** with objects all us-quantales and with morphism (the so-called us-quantale morphisms) all s-quantale morphisms preserving the unit (Rodabaugh, 2007). Now we have enough information to introduce fuzzy semi-quantales:

**Definition 2.** *Let $(L,\leq,\otimes)$ and $(M,\leq,*)$ be s-quantales.*

*(i) An M-fuzzy semi-quantale on L is a map $\mu : L\rightarrow M$ satisfying the following conditions: For all $x,y\in L$ and $\left\{x_j \mid j\in J\right\}\subseteq L$,*

*(FSQ1)* $\mu(x)*\mu(y)\leq\mu(x\otimes y)$,

*(FSQ2)* $\bigwedge_{j\in J}\mu(x_j)\leq\mu\left(\bigvee_{j\in J}x_j\right)$.

*(ii) An M-fuzzy semi-quantale $\mu$ is called strong if $\mu(\top_L)=\top_M$.*

*(iii) In case $(L,\leq,\otimes)$ is a us-quantale with the unit $e_L$, an M-fuzzy semi-quantale $\mu$ is called unital if $\mu(e_L)=\top_M$.*

Throughout this paper, we use the abbreviations $M$-fs-quantale, $M$-fss-quantale and $M$-fus-quantale for $M$-fuzzy semi-quantale, strong $M$-fuzzy semi-quantale and unital $M$-fuzzy semi-quantale, respectively.

**Example 3.** *For $a,b\in\mathbb{R}$ with $a<b$ and for any binary operation $\star$ on $[a,b]$, let $[a,b]_\star$ denote $([a,b],\leq,\star)$, where $\leq$ is the usual ordering. Further, let $\otimes$ and $*$ be binary operations on $[0,1]$ such that $x*y\leq x\otimes y$ for all $x,y\in[0,1]$.*

*(i) The identity map $id_{[0,1]}$ on $[0,1]$ is a $[0,1]_*$-fs-quantale on $[0,1]_\otimes$. In particular, if $\otimes$ is a t-norm T (Klement, Mesiar and Pap, 2000), then $id_{[0,1]}$ is also a $[0,1]_*$-fus-quantale on $[0,1]_T$.*

*(ii) If $\otimes$ is the product t-norm $T_P$ (Klement, Mesiar and Pap, 2000), then for every nonnegative integer n, the function $[0,1]\rightarrow[0,1]$, $x\mapsto x^n$, is a $[0,1]_*$-fus-quantale on $[0,1]_{T_P}$.*

*(iii) For $c,d\in\mathbb{R}$ with $c<d$, let the binary operation $\otimes_1$ on $[a,b]$ and the binary operations $*_1$, $*_2$ on $[c,d]$ be defined by $x\otimes_1 y=\frac{x+y}{2}$, $z*_1 w=\frac{z+w}{2}$ and $z*_1 w=min\{z,w\}$.*
*The linear function $[a,b]\rightarrow[c,d]$, $x\mapsto mx+n$, is a $[c,d]_{*_i}$-fss-quantale on $[a,b]_{\otimes_1}$, where $i=1,2$,*
*$m=\frac{d-c}{b-a}$ and $n=\frac{b.c-d.a}{b-a}$.*

**Definition 4.** *Let M be an s-quantale.*

*(i) M-**FSQuant** is a category that has as objects all pairs $(A,\mu)$ for which A is an s-quantale and $\mu$ is an M-fs-quantale on A, as morphisms from $(A_1,\mu_1)$ to $(A_2,\mu_2)$ all s-quantale morphisms $h : A_1\rightarrow A_2$ such that $\mu_1(x)\leq\mu_2(h(x))$. Composition and identities are the same as in the category **Set** of sets and functions.*

*(ii) M-**FSSQuant** is a non-full subcategory of M-**FSQuant** in which each object $(A,\mu)$ additionally satisfies the property that $\mu$ is strong, and each morphism is additionally strong.*

*(iii) M-**FUSQuant** is a non-full subcategory of M-**FSQuant** in which each object $(A,\mu)$ additionally satisfies the property that A is a us-quantale and $\mu$ is unital, and each morphism is additionally unital.*

# 3 $(L,M)$-QUASI-FUZZY TOPOLOGICAL SPACES

**Definition 5.** *Let $(L,\leq,\otimes)$ and $(M,\leq,*)$ be s-quantales, and $X$ a set.*

*(i) A map $\tau : L^X \to M$ is called an $(L,M)$-quasi-fuzzy topology on $X$ iff $\tau$ is an $M$-fs-quantale on $L^X$, i.e. the next conditions are satisfied for all $f,g \in L^X$ and $\{f_j \mid j \in J\} \subseteq L^X$:*

*(QT1)* $\tau(f) * \tau(g) \leq \tau(f \otimes g)$,

*(QT2)* $\bigwedge_{j\in J} \tau(f_j) \leq \tau\left(\bigvee_{j\in J} f_j\right)$.

*(ii) An $(L,M)$-quasi-fuzzy topology is strong iff $\tau(\top_{L^X}) = \top_M$, where $\top_{L^X} : X \to L$ is the constant map with value $\top_L$.*

*(iii) Let $L$ be a us-quantale with unit $e$. An $(L,M)$-quasi-fuzzy topology is then called an $(L,M)$-fuzzy topology iff $\tau(e_{L^X}) = \top_M$, where $e_{L^X} : X \to L$ is the constant map with value $e$.*

*(iv) $(X,\tau)$ is called an $(L,M)$-quasi-fuzzy (resp. strong $(L,M)$-quasi-fuzzy, $(L,M)$-fuzzy) topological space if $\tau$ is an $(L,M)$-quasi-fuzzy (resp. strong $(L,M)$-quasi-fuzzy, $(L,M)$-fuzzy) topology on $X$.*

**Definition 6.** *Let $(L,\leq,\otimes)$ and $(M,\leq,*)$ be s-quantales.*

*(i) $(L,M)$-**QFTop** denotes a category whose objects are all $(L,M)$-quasi-fuzzy topological spaces, and whose morphisms from $(X,\tau)$ to $(Y,\nu)$ are all functions $f : X \to Y$ such that $\nu \leq \tau \circ f_L^{\leftarrow}$, where $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(G) = G \circ f$. Composition and identities in $(L,M)$-**QFTop** are the same as in **Set**.*

*(ii) $(L,M)$-**SQFTop** is a full subcategory of $(L,M)$-**QFTop** consisting of all strong $(L,M)$-quasi-fuzzy topological spaces.*

*(iii) For a us-quantale $L$, $(L,M)$-**FTop** is a full subcategory of $(L,M)$-**QFTop** consisting of all $(L,M)$-fuzzy topological spaces.*

In case $L$ is an st-s-quantale, $(L,M)$-**SQFTop** coincides with $(L,M)$-**FTop**. As is explained in the example below, our motivation and the necessity of $(L,M)$-(quasi-)fuzzy topological spaces come from the need for a unification of $(L,M)$-fuzzy topological spaces in (Kubiak and Šostak, 2009), $L$-(quasi-)fuzzy topological spaces in (Rodabaugh, 2007) and $L$-fuzzy topological spaces in (Höhle and Šostak, 1999).

**Example 7.** *(i) Let $(L,\leq,\otimes)$ and $(M,\leq,*)$ be s-quantales chosen such as $\otimes = \wedge$, $* = \wedge$ and $(M,\leq)$ is completely distributive, i.e. the identity*

$$\bigwedge_{i\in I}\left(\bigvee_{j\in J_i} c_{ij}\right) = \bigvee_{k\in\prod_{i\in I} J_i}\left(\bigwedge_{i\in I} c_{ik(i)}\right)$$

*holds for every $i \in I$ and $\{c_{ij} \mid j \in J_i\} \subseteq M$. In this case, $(L,M)$-**FTop** (= $(L,M)$-**SQFTop**) is the category **TOP**$(L,M)$ of $(L,M)$-fuzzy topological spaces in (Kubiak and Šostak, 2009).*

*(ii) For any s-quantale $L$, $(L,L)$-**QFTop** is the category $L$-**QFTop** of $L$-quasi-fuzzy topological spaces in (Rodabaugh, 2007).*

*(iii) For any st-s-quantale $L$, $(L,L)$-**FTop** is the category $L$-**FTop** of $L$-fuzzy topological spaces in (Rodabaugh, 2007).*

*(iv) For any cqm-lattice $L$, $(L,L)$-**SQFTop** is the category $L$-**FTOP** of $L$-fuzzy topological spaces in (Höhle and Šostak, 1999).*

# 4 RELATIONS BETWEEN $M$-FSQuant AND $(L,M)$-QFTop

The main objective of this section is to establish a dual adjunction between $(L,M)$-**QFTop** and $M$-**FSQuant** and then to show that this dual adjunction turns into a dual equivalence between the full subcategory of $M$-**FSQuant** of $(L,M)$-spatial objects and the full subcategory of $(L,M)$-**QFTop** of $(L,M)$-sober objects. In order to realize our aim, we first recall some well-known category-theoretic tools in the next subsection. Those already experienced with categories can skip this subsection. For more details and for all other category-theoretic notions not explicitly stated in this paper, we refer the reader to (Adámek, Herrlich and Strecker, 1990).

## 4.1 Adjoint Situations and Equivalences

Our main results in this paper are formulated on the basis of the notions of adjoint situation, equivalence and opposite (dual) category. By definition, an adjoint situation $(\rho,\phi) : F \dashv G : \mathbf{C} \to \mathbf{D}$ consists of functors $G : \mathbf{C} \to \mathbf{D}$, $F : \mathbf{D} \to \mathbf{C}$, and natural transformations $id_{\mathbf{D}} \xrightarrow{\rho} GF$ (called the unit) and $FG \xrightarrow{\phi} id_{\mathbf{C}}$ (called the co-unit) satisfying the adjunction identities

$$G(\phi_A) \circ \rho_{G(A)} = id_{G(A)} \text{ and } \phi_{F(B)} \circ F(\rho_B) = id_{F(B)}$$

for all $A$ in $\mathbf{C}$ and $B$ in $\mathbf{D}$. If $(\rho,\phi) : F \dashv G : \mathbf{C} \to \mathbf{D}$ is an adjoint situation for some $\rho$ and $\phi$, then $F$ is said to be a left adjoint to $G$, $F \dashv G$ in symbols. A functor $G : \mathbf{C} \to \mathbf{D}$ is called an equivalence if it is full, faithful and isomorphism-dense. In this case, $\mathbf{C}$ and $\mathbf{D}$ are called equivalent categories, denoted by $\mathbf{C} \sim \mathbf{D}$. Equivalences can also be stated in terms of adjoint situations: $\mathbf{C} \sim \mathbf{D}$ iff there exists an adjoint situation $(\rho,\phi) : F \dashv G : \mathbf{C} \to \mathbf{D}$ with natural isomorphisms $\rho$ and $\phi$.

The opposite (dual) of a category **C** is defined as a category $\mathbf{C}^{op}$, whose objects are the same as **C**-objects, but morphisms $A \stackrel{u^{op}}{\to} B$ are **C**-morphisms $B \stackrel{u}{\to} A$ in the opposite direction. Composition of $\mathbf{C}^{op}$-morphisms $A \stackrel{u^{op}}{\to} B$ and $B \stackrel{v^{op}}{\to} C$ is given as $A \stackrel{(u\circ v)^{op}}{\to} B$, while identities of **C** and $\mathbf{C}^{op}$ are the same. We say that a category **C** is dually equivalent to another category **D** if $\mathbf{C}^{op} \sim \mathbf{D}$.

**Proposition 8.** *(Porst and Tholen, 1990) Given an adjoint situation* $(\rho,\phi) : F \dashv G : \mathbf{C}^{op} \to \mathbf{D}$, *let* $Fix(\phi)$ *denote the full subcategory of* **C** *of all* **C**-*objects A such that* $\phi_A^{op} : A \to FGA$ *is a* **C**-*isomorphism, and* $Fix(\rho)$ *the full subcategory of* **D** *of all* **D**-*objects B such that* $\rho_B : B \to GFB$ *is a* **D**-*isomorphism. Then the restriction of* $F \dashv G$ *to* $[Fix(\phi)]^{op}$ *and* $Fix(\rho)$ *induces an equivalence* $[Fix(\phi)]^{op} \sim Fix(\rho)$.

## 4.2 Adjunction Between $M$-FSQuant$^{op}$ and $(L,M)$-QFTop

The map $S : (L,M)$-**QFTop** $\to M$-**FSQuant**$^{op}$, defined by

$$S\Big((X,\tau) \stackrel{f}{\longrightarrow} (Y,\nu)\Big) = \left(L^X,\tau\right) \stackrel{\left(f_L^{\leftarrow}\right)^{op}}{\longrightarrow} \left(L^Y,\nu\right),$$

is clearly a functor. We now construct a functor $T : M$-**FSQuant**$^{op} \to (L,M)$-**QFTop** with the property that $S \dashv T$ by making use of the next lemmas.

**Lemma 9.** *Let* $(M,\leq,*)$ *be a quantale such that*

$$\perp_M * \top_M = \top_M * \perp_M = \perp_M$$

*and* $(M,\leq)$ *is completely distributive.*

*If* $h : (A_1,\mu_1) \to (A_2,\mu_2)$ *is an M-**FSQuant**-morphism, then* $h_M^{\to}(\mu_1)$ *is an M-fs-quantale on* $A_2$, *where* $h_M^{\to} : M^{A_1} \to M^{A_2}$ *is defined by*

$$h_M^{\to}(\alpha)(y) = \bigvee_{y=h(x)} \alpha(x), \ \forall \alpha \in M^{A_1}, \ \forall y \in A_2.$$

*Proof.* Let $\otimes_i$ denote the tensor product on $A_i$ for $i = 1,2$. In order to see (FSQ1), let us pick arbitrary $y_1, y_2 \in A_2$, and say $\gamma = h_M^{\to}(\mu_1)(y_1) * h_M^{\to}(\mu_1)(y_2)$. If $h^{-1}(y_1) = \emptyset$ or $h^{-1}(y_2) = \emptyset$, then

$$\gamma = \perp_M \leq h_M^{\to}(\mu_1)(y_1 \otimes_2 y_2).$$

Suppose $h^{-1}(y_1) \neq \emptyset$ and $h^{-1}(y_2) \neq \emptyset$. By using the distributivity of $*$ over arbitrary $\bigvee$,

$$\begin{aligned}
\gamma &= \left(\bigvee_{y_1=h(x_1)} \mu_1(x_1)\right) * \left(\bigvee_{y_2=h(x_2)} \mu_1(x_2)\right) \\
&= \bigvee_{y_1=h(x_1), y_2=h(x_1)} \mu_1(x_1) * \mu_1(x_2) \\
&\leq \bigvee_{y_1=h(x_1), y_2=h(x_1)} \mu_1(x_1 \otimes_1 x_2) \\
&\leq \bigvee_{y_1 \otimes_2 y_2 = h(z)} \mu_1(z) = h_M^{\to}(\mu_1)(y_1 \otimes_2 y_2).
\end{aligned}$$

To prove (FSQ2), let us take an arbitrary $\left\{y_j \mid j \in J\right\} \subseteq A_2$. By the complete distributivity of $(M,\leq)$, we may write

$$\begin{aligned}
\bigwedge_{j\in J} h_M^{\to}(\mu_1)(y_j) &= \bigwedge_{j\in J}\left(\bigvee_{x\in h^{-1}(y_j)} \mu_1(x)\right) \\
&= \bigvee_{k \in \prod_{j\in J} h^{-1}(y_j)} \left(\bigwedge_{j\in J} \mu_1(k(j))\right) \\
&\leq \bigvee_{k \in \prod_{j\in J} h^{-1}(y_j)} \left(\mu_1\left(\bigvee_{j\in J} k(j)\right)\right).
\end{aligned}$$

Furthermore, for every $k \in \prod_{j\in J} h^{-1}(y_j)$, since $h(k(j)) = y_j$ for every $j \in J$, we easily get

$$\bigvee_{j\in J} k(j) \in h^{-1}\left(\bigvee_{j\in J} y_j\right),$$

which gives

$$\begin{aligned}
\bigvee_{k \in \prod_{j\in J} h^{-1}(y_j)} \left(\mu_1\left(\bigvee_{j\in J} k(j)\right)\right) &\leq \bigvee_{z\in h^{-1}\left(\bigvee_{j\in J} y_j\right)} \mu_1(z) \\
&= h_M^{\to}(\mu_1)\left(\bigvee_{j\in J} y_j\right).
\end{aligned}$$

Therefore, it follows that

$$\bigwedge_{j\in J} h_M^{\to}(\mu_1)(y_j) \leq h_M^{\to}(\mu_1)\left(\bigvee_{j\in J} y_j\right),$$

and so does the assertion. □

**Lemma 10.** *Let L be an s-quantale, and M a quantale with the properties in Lemma 9. For every s-quantale A, denote by* $St(A)$ *the set of all s-quantale morphisms from A to L, and by* $\langle A\rangle$ *the evaluation map* $A \to L^{St(A)}$, *i.e.*

$$\langle A\rangle(a)(h) = h(a), \ \forall a \in A, \ \forall h \in St(A).$$

*Then* $\left(St(A), \tau_{(A,\mu)}\right)$ *is an* $(L,M)$-***QFTop****-object, where* $\tau_{(A,\mu)} = \langle A\rangle_M^{\rightarrow}(\mu)$.

*Proof.* Consider the constant map $\theta : L^{St(A)} \to M$ with value $\top_M$. Since $\langle A\rangle : (A,\mu) \to \left(L^{St(A)}, \theta\right)$ is an $M$-**FSQuant**-morphism, we directly obtain from Lemma 9 that $\tau_{(A,\mu)}$ is an $M$-fs-quantale on $L^{St(A)}$, i.e. $\left(St(A), \tau_{(A,\mu)}\right)$ is an $(L,M)$-**QFTop**-object. □

In the remainder of this paper, we always assume that $L$ is an s-quantale and $M$ is a quantale with the properties in Lemma 9.

**Theorem 11.** $T : M$-***FSQuant***$^{op} \to (L,M)$-***QFTop****, defined by*

$$T\left((A_1,\mu_1) \xrightarrow{u} (A_2,\mu_2)\right) = T(A_1,\mu_1) \xrightarrow{T(u)} T(A_2,\mu_2),$$

*is a functor, where* $T(A,\mu) = \left(St(A), \tau_{(A,\mu)}\right)$ *and* $T(u) : St(A_1) \to St(A_2)$ *is a function, defined by*

$$T(u)(h) = h \circ u^{op}, \ \forall h \in St(A_1).$$

*Proof.* By Lemma 10, $T$ is an object function from $M$-**FSQuant**$^{op}$ to $(L,M)$-**QFTop**. Since $T$ obviously preserves composition and identities, we only need to see that $T$ is a morphism function, i.e. for every $M$-**FSQuant**$^{op}$-morphism $(A_1,\mu_1) \xrightarrow{u} (A_2,\mu_2)$, $T(u) : T(A_1,\mu_1) \to T(A_2,\mu_2)$ is an $(L,M)$-**QFTop**-morphism, this means that, the inequality

$$\tau_{(A_2,\mu_2)}(G) \leq \tau_{(A_1,\mu_1)}\left(T(u)_L^{\leftarrow}(G)\right) \qquad (1)$$

holds for every $G \in L^{St(A_2)}$. In order to prove (1), we first show that for every $a_2 \in A_2$ with $G = \langle A_2\rangle(a_2)$,

$$T(u)_L^{\leftarrow}(G) = \langle A_1\rangle\left(u^{op}(a_2)\right). \qquad (2)$$

Given $a_2 \in A_2$, suppose $G = \langle A_2\rangle(a_2)$, i.e. for every $h_2 \in St(A_2)$,

$$G(h_2) = \langle A_2\rangle(a_2)(h_2) = h_2(a_2).$$

Then, for every $h_1 \in St(A_1)$, since

$$T(u)(h_1) = h_1 \circ u^{op} \in St(A_2),$$

and by the definition of $T(u)_L^{\leftarrow}$, we obtain (2) from the following observation:

$$\begin{aligned}
\left[T(u)_L^{\leftarrow}(G)\right](h_1) &= G(T(u)(h_1)) \\
&= [T(u)(h_1)](a_2) \\
&= h_1 \circ u^{op}(a_2) = h_1\left(u^{op}(a_2)\right) \\
&= \left[\langle A_1\rangle\left(u^{op}(a_2)\right)\right](h_1).
\end{aligned}$$

On the other hand, since $(A_2,\mu_2) \xrightarrow{u^{op}} (A_1,\mu_1)$ is an $M$-**FSQuant**-morphism, we also have

$$\mu_2(a_2) \leq \mu_1\left(u^{op}(a_2)\right).$$

This inequality together with (2) allow us to put down

$$\begin{aligned}
\tau_{(A_2,\mu_2)}(G) &= \bigvee_{G=\langle A_2\rangle(a_2)} \mu_2(a_2) \\
&\leq \bigvee_{T(u)_L^{\leftarrow}(G)=\langle A_1\rangle(u^{op}(a_2))} \mu_2(a_2) \\
&\leq \bigvee_{T(u)_L^{\leftarrow}(G)=\langle A_1\rangle(u^{op}(a_2))} \mu_1\left(u^{op}(a_2)\right) \\
&\leq \bigvee_{T(u)_L^{\leftarrow}(G)=\langle A_1\rangle(a_1)} \mu_1(a_1) \\
&= \tau_{(A_1,\mu_1)}\left(T(u)_L^{\leftarrow}(G)\right).
\end{aligned}$$

□

**Proposition 12.** *S is left adjoint to T.*

*Proof.* For every $(L,M)$-**QFTop**-object $(X,\tau)$, define the function $\eta_{(X,\tau)} : X \to St\left(L^X\right)$ by $\eta_{(X,\tau)}(x) = \pi_x$, where $\pi_x : L^X \to L$ is the $x$-th projection function for every $x \in X$.

One can easily see that for every $(L,M)$-**QFTop**-object $(X,\tau)$ and every $M$-**FSQuant**-object $(A,\mu)$,

$$\begin{aligned}
\eta_{(X,\tau)} &: (X,\tau) \to TS(X,\tau) \text{ and} \\
\varepsilon_{(A,\mu)} &= \langle A\rangle^{op} : ST(A,\mu) \to (A,\mu)
\end{aligned}$$

are, respectively, an $(L,M)$-**QFTop**-morphism and an $M$-**FSQuant**$^{op}$-morphism. Moreover,

$$\begin{aligned}
\eta &= \left(\eta_{(X,\tau)}\right) : id_{(L,M)\text{-}\mathbf{QFTop}} \to TS \text{ and} \\
\varepsilon &= \left(\varepsilon_{(A,\mu)}\right) : ST \to id_{M\text{-}\mathbf{FSQuant}^{op}}
\end{aligned}$$

are natural transformations making

$$(\eta,\varepsilon) : S \dashv T : M\text{-}\mathbf{FSQuant}^{op} \to (L,M)\text{-}\mathbf{QFTop}$$

an adjoint situation, and hence $S \dashv T$. □

### 4.3 Duality between $(L,M)$-Spatiality and $(L,M)$-Sobriety

We concluded the preceding subsection with the adjoint situation

$$(\eta,\varepsilon) : S \dashv T : M\text{-}\mathbf{FSQuant}^{op} \to (L,M)\text{-}\mathbf{QFTop}.$$

With the help of Proposition 8, we restrict, in this subsection, this adjoint situation to an equivalence of subcategories of $M$-**FSQuant**$^{op}$ and $(L,M)$-**QFTop** involving reciprocal notions of spatiality and sobriety, which are introduced as follows.

**Definition 13.** *(i) An* $M$-***FSQuant****-object* $(A,\mu)$ *is called* $(L,M)$*-spatial iff* $\langle A\rangle : (A,\mu) \to ST(A,\mu)$ *is an* $M$-***FSQuant****-isomorphism.*

*(ii) An* $(L,M)$-***QFTop****-object* $(X,\tau)$ *is called* $(L,M)$*-sober iff* $\eta_{(X,\tau)} : (X,\tau) \to TS(X,\tau)$ *is an* $(L,M)$-***QFTop****-isomorphism.*

$(L,M)$-spatiality and $(L,M)$-sobriety can also be stated in the following more explicit form:

**Proposition 14.** *(i) An $M$-**FSQuant**-object $(A,\mu)$ is $(L,M)$-spatial iff the evaluation map $\langle A\rangle : A \to L^{St(A)}$ is a bijection, i.e. for every $t \in L^{St(A)}$, there exists a unique $a \in A$ such that $t = \langle A\rangle (a)$.*

*(ii) An $(L,M)$-**QFTop**-object $(X,\tau)$ is $(L,M)$-sober iff the projection morphisms $\pi_x : L^X \to L$ are the only s-quantale morphisms from $L^X$ to L, i.e. for every $w \in St\left(L^X\right)$, there exists a unique $x \in X$ such that $t = \pi_x$.*

**Corollary 15.** *The full subcategory of $M$-**FSQuant** of $(L,M)$-spatial objects is dually equivalent to the full subcategory of $(L,M)$-**QFTop** of $(L,M)$-sober objects.*

# 5 DUALS of $M$-FSSQuant and $M$-FUSQuant

The relationships in Section 4 can also be formulated for the categories $M$-**FSSQuant** and $M$-**FUSQuant** by making some slight changes in the subsections 4.2 and 4.3. To explain how this can be done, we first fix some notations. For every s-quantale (resp. us-quantale) $A$, we denote by $St_1(A)$ (resp. $St_2(A)$) the set of all strong (resp. unital) s-quantale morphisms from $A$ to $L$, where $L$ is assumed to be a us-quantale in case $A$ is a us-quantale, and by $\langle A\rangle_i$ the evaluation map $A \to L^{St_i(A)}$, i.e.

$$\langle A\rangle_i (a)(h) = h(a),\ \forall a \in A,\ \forall h \in St_i(A)$$

for $i = 1,2$.

It is clear that the restriction of the functor

$$S : (L,M)\textbf{-QFTop} \to M\textbf{-FSQuant}^{op}$$

to $(L,M)$-**SQFTop** gives a functor

$$S_1 : (L,M)\textbf{-SQFTop} \to M\textbf{-FSSQuant}^{op}.$$

Likewise, for a us-quantale $L$, $S$ can be restricted to a functor $S_2 : (L,M)$-**FTop** $\to M$-**FUSQuant**$^{op}$. In the reverse direction, by following the same steps in Theorem 11, we may define functors

$$\begin{aligned} T_1 &: M\textbf{-FSSQuant}^{op} \to (L,M)\textbf{-SQFTop} \text{ and} \\ T_2 &: M\textbf{-FUSQuant}^{op} \to (L,M)\textbf{-FTop} \end{aligned}$$

by

$$T_i\left((A,\mu) \xrightarrow{u} (B,\lambda)\right) = T_i(A,\mu) \xrightarrow{T_i(u)} T_i(B,\lambda),$$

where $i = 1,2$, $L$ is a us-quantale for $i = 2$,

$$T_i(A,\mu) = \left(St_i(A), \tau^{(i)}_{(A,\mu)}\right),\ \tau^{(i)}_{(A,\mu)} = \left(\langle A\rangle_i\right)^{\rightarrow}_M(\mu)$$

and $T_i(u) : St_i(A) \to St_i(B)$ is a function given by

$$T_i(u)(h) = h \circ u^{op},\ \forall h \in St_i(A).$$

An analogue of Proposition 12 can be expressed for the functors $S_i$ and $T_i$: $S_i \dashv T_i$ $(i = 1,2)$. The unit $\eta^i$ and co-unit $\varepsilon^i$ of the adjunction $S_i \dashv T_i$ are the families $\eta^i = \left(\eta^i_{(X,\tau)}\right)$ and $\varepsilon^i = \left(\varepsilon^i_{(A,\mu)}\right)$, where $i = 1,2$,

$$\begin{aligned} \eta^i_{(X,\tau)} &: (X,\tau) \to T_iS_i(X,\tau) \text{ and} \\ \varepsilon^i_{(A,\mu)} &: S_iT_i(A,\mu) \to (A,\mu) \end{aligned}$$

are given by

$$\eta^i_{(X,\tau)}(x) = \pi_x \text{ and } \left(\varepsilon^i_{(A,\mu)}\right)^{op} = \langle A\rangle_i.$$

Consequently, $(\eta^i,\varepsilon^i) : S_i \dashv T_i : \mathbf{C}_i^{op} \to \mathbf{D}_i$ is an adjoint situation, where $i = 1,2$, $L$ is a us-quantale for $i = 2$,

$$\begin{aligned} \mathbf{C}_1 &= M\textbf{-FSSQuant}, \mathbf{D}_1 = (L,M)\textbf{-SQFTop}, \\ \mathbf{C}_2 &= M\textbf{-FUSQuant}, \mathbf{D}_2 = (L,M)\textbf{-FTop}. \end{aligned}$$

In a similar fashion to Corollary 15, the adjunction $S_i \dashv T_i$ restricts to an equivalence on the basis of the following notions of spatiality and sobriety:

**Definition 16.** *(i) An $M$-**FSSQuant**-object $(A,\mu)$ is called $(L,M)$-s-spatial iff $\langle A\rangle_1 : (A,\mu) \to S_1T_1(A,\mu)$ is an $M$-**FSSQuant**-isomorphism.*

*(ii) An $(L,M)$-**SQFTop**-object $(X,\tau)$ is called $(L,M)$-s-sober iff $\eta^1_{(X,\tau)} : (X,\tau) \to T_1S_1(X,\tau)$ is an $(L,M)$-**SQFTop**-isomorphism.*

**Definition 17.** *Let L be a us-quantale.*

*(i) An $M$-**FUSQuant**-object $(A,\mu)$ is called $(L,M)$-us-spatial iff $\langle A\rangle_2 : (A,\mu) \to S_2T_2(A,\mu)$ is an $M$-**FUSQuant**-isomorphism.*

*(ii) An $(L,M)$-**FTop**-object $(X,\tau)$ is called $(L,M)$-us-sober iff $\eta^2_{(X,\tau)} : (X,\tau) \to T_2S_2(X,\tau)$ is an $(L,M)$-**FTop**-isomorphism.*

**Corollary 18.** *(i) The full subcategory of $M$-**FSSQuant** of $(L,M)$-s-spatial objects is dually equivalent to the full subcategory of $(L,M)$-**SQFTop** of $(L,M)$-s-sober objects.*

*(ii) For any us-quantale L, the full subcategory of $M$-**FUSQuant** of $(L,M)$-us-spatial objects is dually equivalent to the full subcategory of $(L,M)$-**FTop** of $(L,M)$-us-sober objects.*

# 6 CONCLUSIONS

In this paper, we have introduced $M$-fuzzy semi-quantales and $(L,M)$-quasi-fuzzy topological spaces. The former here fuzzify semi-quantales of Rodabaugh (Rodabaugh, 2007), while the latter unify $(L,M)$-fuzzy topological spaces of Kubiak and Sostak (Kubiak and Šostak, 2009), $L$-quasi-fuzzy topological spaces of Rodabaugh (Rodabaugh, 2007) and $L$-fuzzy topological spaces of Höhle and Šostak (Höhle and Šostak, 1999). We then have constructed their categories that enable us to extend the famous Papert-Papert-Isbell adjunction (Isbell, 1972; Jhonstone, 1986; Papert and Papert, 1957/1958) to the present setting. One of the central results of this paper is that there exists a dual adjunction between the category of $(L,M)$-quasi-fuzzy topological spaces and the category of $M$-fuzzy semi-quantales. As the other one, it is shown that this adjunction gives rise to a dual equivalence between the category of $(L,M)$-sober $(L,M)$-quasi-fuzzy topological spaces and the category of $(L,M)$-spatial $M$-fuzzy semi-quantales. We finally demonstrate that analogues of these results are also valid for the categories of strong $M$-fuzzy semi-quantales and of unital $M$-fuzzy semi-quantales.

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