Current Trends in Mathematical Image Analysis
A Survey

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Abstract: The main task of the survey is to explain and discuss the opportunities and limitations of algebraic approaches in image analysis. During recent years there was accepted that algebraic techniques, in particular different kinds of image algebras, is the most prospective direction of construction of the mathematical theory of image analysis and of development an universal algebraic language for representing image analysis transforms and image models. The main goal of the Algebraic Approach is designing of a unified scheme for representation of objects under recognition and its transforms in the form of certain algebraic structures. It makes possible to develop corresponding regular structures ready for analysis by algebraic, geometrical and topological techniques. Development of this line of image analysis and pattern recognition is of crucial importance for automatic image-mining and application problems solving, in particular for diversification classes and types of solvable problems and for essential increasing of solution efficiency and quality.

1 INTRODUCTION

The specificity, complexity and difficulties of image analysis and estimation (IAE) problems stem from necessity to achieve some balance between such highly contradictory factors as goals and tasks of a problem solving, the nature of visual perception, ways and means of an image acquisition, formation, reproduction and rendering, and mathematical, computational and technological means allowable for the IAE.

During recent years there was accepted that algebraic techniques, in particular different kinds of image algebras, is the most prospective direction of construction of the mathematical theory of image analysis and of development of an universal algebraic language for representing image analysis transforms and image models.

Development of this line of image analysis and pattern recognition is of crucial importance for automatic image-mining and application problems solving, in particular for diversification classes and types of solvable problems and for essential increasing of solution efficiency and quality.

Images are one of the main tools to represent and transfer information needed to automate the intellectual decision-making in many application areas. Increasing the efficiency, including automatization, of gathering information from images can help increase the efficiency of intellectual decision-making.

Recently, this part of image analysis called image mining in English publications has been often set off into a separate line of research.

We list the functions of particular aspects of image handling. Image processing and analysis provides for image mining, which is necessary for decision-making, while the very decision-making is done by methods of mathematical theory of pattern recognition. To link these two stages, the information gathered from the image after it is analysed is transformed so that standard recognition algorithms could process it. Note that although this stage seems to have an “intermediate” character, it is the fundamental and necessary condition for the overall recognition to be feasible.

At present, automated image mining is the main strategic goal of fundamental research in image analysis, recognition and understanding and development of the proper information technology and algorithmic software systems.

To ensure such automatization, we need to develop and evolve a new approach to analysing and evaluating information represented in the form of images. To do it, the “Algebraic Approach” of Yu. I.

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Zhuravlev (Zhuravlev, 1998) was modified for the case when the initial information is represented in the form of images. The result is the descriptive approach to image analysis and understanding (DA) proposed and justified by I. B. Gurevich and developed by his pupils (Gurevich, 1989, 1991, 2005), (Gurevich, Jernova, 2003), (Gurevich, Koryabkina, 2006), (Gurevich, Yashina, 2006, 2008, 2012), (Gurevich et al., 2008).

By now, image analysis and evaluation have a wide experience gained in applying mathematical methods from different sections of mathematics, computer science and physics, in particular algebra, geometry, discrete mathematics, mathematical logic, probability theory, mathematical statistics, mathematical analysis, and mathematical theory of pattern recognition, digital signal processing, and optics.

On the other hand, with all this diversity of applied methods, we still need to have a regular basis to arrange and choose suitable methods of image analysis, represent, in an unified way, the processed data (images), meeting the requirements standard recognition algorithms impose on initial information, construct mathematical models of images designed for recognition problems, and, on the whole, establish the universal language for unified description of images and transformations over them.

In 1970s, Yu. I. Zhuravlev proposed the so-called "Algebraic Approach to Recognition and Classification Problems" (Zhuravlev, 1998), where he defined formalization methods for describing heuristic algorithms of pattern recognition and proposed the universal structure of recognition algorithms. In the same years, U. Grenander stated his "Pattern Theory" (Grenander, 1976, 1978, 1981, 1993, 1996), where he considered methods of data representation and transformation in recognition problems in terms of regular combinatorial structures, leveraging algebraic and probabilistic apparatus. Both approaches dealt with the recognition problem in its classical statement and did not touch upon representation of initial data in the form of images.

Then, up to the middle of 1990s, there was a slight drop in the interest in descriptive and algebraic aspects in pattern recognition and image analysis.

By the middle of 1990s, it became obvious that for the development of image analysis and recognition, it is critical to: understand the nature of the initial information – images, find methods of image representation and description that allow constructing image models designed for recognition problems, establish the mathematical language designed for unified description of image models and their transformations that allow constructing image models and solving recognition problems, and construct models to solve recognition problems in the form of standard algorithmic schemes that allow, in the general case, moving from the initial image to its model and from the model to the sought solution.

The DA gives a single conceptual structure that helps develop and implement these models and the mathematical language (Gurevich, 1989, 1991). The main DA purpose is to structure and standardize different methods, operations and representations used in image recognition and analysis. The DA provides the conceptual and mathematical basis for image mining, with its axiomatic and formal configurations giving the ways and tools to represent and describe images to be analysed and evaluated.

In this work, we give a brief review of the main algebraic methods and features. The work consists of seven main sections (along with Introduction and Conclusions).

"State of the art of mathematical theory of image analysis" is the section that describes modern trends in developing of mathematical tools for automation of image analysis, in particular in image mining.

The section "Steps of the algebraization" presents leading approaches of mathematical theory for image analysis oriented for automation of image analysis and understanding.

The section "The algebraic approach to recognition classification and forecasting problems by Zhuravlev" contains main aspects of algebraic theory of Yu.I.Zhuravlev.

The section "Image Algebras" consists of brief description of different image algebras.

The section "Descriptive approach to image analysis" presents a methodology, mathematical and computational techniques for automation of image mining on the base of Descriptive Approach to Image Analysis.

In conclusion, there are some words about opportunities of algebraic techniques via an example of biomedical image analysis practical problem and discussion the prospects of the mathematical image analysis development.

2 STATE OF THE ART OF MATHEMATICAL THEORY OF IMAGE ANALYSIS

To automate image mining, we need an integrated approach to leverage the potential of mathematical apparatus of the main lines in transforming and analysing information represented in the form of images, viz. image processing, analysis, recognition and understanding.

Done by pattern recognition methods, image mining now tends to multiplicity (multialgorithmic and multimodel) and fusion of the results, i.e., several different algorithms are applied in parallel to process the same model and several different models of the same initial data to solve the problem and then the results are fused to obtain the most accurate solution.

Multialgorithmic classifiers and multimodel and multiple-aspect image representations are the common tools to implement this multiplicity and fusion. Note that it was Yu. I. Zhuravlev who obtained the first and fundamental results in this area in 1970s (Zhuravlev, 1998).

From 1970s, the most part of image recognition applications and considerable part of research in artificial intelligence deal with images. As a result, new technical tools emerged to obtain information that allow representing recorded and accumulated data in the form of images and the image recognition itself became more popular as the powerful and efficient methodology to process, analyse data mathematically, and detect hidden regularities. Various scientific and technical, economic and social factors make the application domain of image recognition experience grow constantly.

There are internal scientific problems that have arisen within image recognition. First, these imply algebraizing the image recognition theory, arranging image recognition algorithms, estimating the algorithmic complexity of the image recognition problem, automating the synthesis of the corresponding efficient procedures, formalizing the description of the image as the recognition object, making the choice of the system of representations of the image in the recognition process regular, and some others. It is the problems that form the basis of the mathematical agenda of the descriptive theory of image recognition developed using the ideas of the algebraic approach to recognition (Zhuravlev, 1998) to create a systematized set of methods and tools of data processing in image recognition and analysis problems.

There are three main issues one need to solve when dealing with images–describe (simulate) images; develop, study and optimize the selection of mathematical methods and tools of data processing in image recognition; and implement mathematical methods of image analysis on a software and hardware basis.

What makes image analysis and recognition problems peculiar, complex and thus difficult and catching is the necessity to find a compromise between rather contradictory factors. These factors are the requirements imposed on the analysis, the nature of visual perception, the ways to obtain, form and reproduce images and the existing mathematical and technical ways to process them. The main contradiction is between the nature of the image and the analysis based on formal description (a model, in essence) of the object, which lies in the fact that to leverage the fact that information is represented in the form of images, it is necessary to make this information non-depictive since the corresponding algorithms can only process certain symbolic descriptions.

Most methods of image processing are purely heuristic, with their quality essentially given by the degree to which they are successful in coping with the “depictive” nature of the image using the “non-depictive” tools, i.e., in employing procedures that do not depend on the fact that the information to be processed is organized in the form of images.

When we solve an image recognition problem, it is very important that we are able to choose the right recognition algorithm in a great number of known algorithms, i.e., we need to choose the best in some sense algorithm in the particular situation. It is obvious that both in image recognition and in solving recognition problems with standard teaching information (Zhuravlev, 1998), to make the choice of the best algorithm systematic, we need to introduce and formalize the corresponding objects of mathematical theory of image recognition, in particular, the concept of image recognition algorithm.

It is known that the necessity to state and solve the problem of choosing the algorithm with respect to the recognition quality functional led to introducing the concept of the model of recognizing algorithm. To choose optimal or acceptable procedure to solve the particular problem, one needed to fix the class of algorithms somehow. This is the first reason that led to the necessity to synthesize models of recognition algorithms.

With the concept of the model of recognizing algorithm, we can apply strict mathematical methods to study the sets of incorrect recognition procedures (i.e., heuristic procedures that are not justified mathematically but were experimentally tested in solving real recognition problems). Analysing the totality of incorrect recognition algorithms as they are accumulated, we can select and describe particular algorithms as well as principles to form them. Acting over subsets of algorithms and first formed in a poorly formalized form, these principles can then become accurate mathematical descriptions. At this stage, principles are chosen on a heuristic basis while algorithms generated according to it can be constructed in a standard way. It is in this sense that formalization of different principles of constructing recognizing algorithms results in models of recognizing algorithms.

To construct the model of recognizing algorithm, we need to describe sets of incorrect procedures that nevertheless are efficient in solving practical problems in a uniform way. To give such set, we specify variables, objects, functions, and parameters and their exact variation area, thus introducing the sought model of the algorithm. Given some set of the corresponding variables, objects, parameters and types of functions, we can single out some fixed algorithm from the model we consider.

To construct the model of an image recognition algorithm and determine the proper class of recognition algorithms, it is not enough to transfer the concept of the model of recognizing algorithm developed in the mathematical recognition theory automatically to the image domain and directly use formal representations of a number of known recognition models studied in classical recognition theory (Zhuravlev, 1998). As noted above, the nature and matter of image recognition problems differ from that of the mathematical recognition theory in its classical statement. When we move from classical recognition problems to image recognition problems, there arise mathematical problems due to formal description of the image as the object to be analysed.

To obtain formal descriptions of images as objects to be analysed and form and choose recognition procedures, we study the internal structure and content of the image as the result of the operations that can be performed to construct it of sub-images and other objects of simpler nature, i.e., primitives and objects singled out on the image during different stages of handling it (depending on the aspect, morphological and/or scale level used to form the image model). Since this way of characterizing the image is operational, we can consider the whole process of image processing and recognition, including construction of formal description –model of the image, as a system of transformations implemented on the image and given on the equivalence classes that represent ensembles of admissible images (Gurevich, Yashina, 2006). Hence, we operate with the hierarchy of formal descriptions of images, i.e., image models used in recognition relate to different aspects and/or morphological (scale) levels of image representation. In essence, these are multiple-aspect and/or multilevel models that allow choosing and changing the necessary degree of detail of description of the recognition object in the course of solving the problem. This approach to formal description of images forms the basis for the multimodel representation of images in recognition problems.

Note that the idea to create a single theory that embraces different approaches and operations used in image and signal processing has a history of its own, with works of von Neumann continued by S. Unger, M. Duff, G. Ritter, J. Serra, S. Sternberg and others (Under, 1958), (Duff et al., 1973), (Ritter, 2001), (Serra, 1982), (Sternberg, 1985) playing an important role in it.

The main stages of algebraization are:

- Mathematical Morphology (G. Matheron, J. Serra [1970's])
- Algorithm Algebra by Yu.I.Zhuravlev (Yu. Zhuravlev [1970's]
- Pattern Theory (U. Grenander [1970's]
- Theory of Categories Techniques in Pattern Recognition (M.Pavel [1970's])
- Image Algebra (Serra, Sternberg [1980's]
- Standard Image Algebra (Ritter [1990's])
- Descriptive Image Algebra (DIA) (Gurevich [1990-2000])
- DIA with one ring (Gurevich, Yashina [2001 to date]).

3 ZHURAVLEV ALGEBRAIC APPROACH

"The Algebraic Approach to Recognition, Classification and Forecasting Problems" (Yu.Zhuravlev) (Zhuravlev, 1998) is mathematical set-up of a pattern recognition problem, correctness and regularity conditions, multiple classifiers.

One of the topical problems in image recognition is searching for an algorithm that would provide a correct classification of an image by its description (i.e. the algorithms that produce zero errors on a control set of objects). The approach to image recognition that is developed by the present authors is a specialization of the algebraic approach to recognition and classification problems originally designed by Yu.I. Zhuravlev (Zhuravlev, 1998). The relational for this approach is the fact that there are no accurate mathematical models for weakly formalized fields such as geology, biology, medicine, and sociology. However, in many cases, inexact methods based on heuristic considerations are practically effective. Therefore, it is sufficient to construct a family of such heuristic algorithms for solving appropriate problems and then construct the algebraic closure of this family. The existence theorem has been proved, which states that any problem among the set of problems associated with the study of poorly formalized situations is solvable in this closure.

Suppose we give a certain set of admissible patterns described by n-dimensional vectors of features. The set of admissible patterns is covered by a finite number of subsets, called classes. Let there exist l classes $K_1, \ldots, K_l$. There is a recognition algorithm A that constructs an l-dimensional information vector by an n-dimensional description vector. Recall that an information vector is the vector of membership of an object in the classes in which the values of elements of the information vector 0, 1, Δ are interpreted, according to (Zhuravlev, 1998), as "an object does not belong to the class," "an object belongs to the class," and "the algorithm cannot determine whether or not an object belongs to the class." We will assume that each recognition algorithm $A \in \{A\}$ can be represented as a sequential execution of algorithms B and C, where B is the recognition operator that transforms learning information and the description of an admissible object into a numerical vector, called the estimate vector, and C is the decision rule that transforms an arbitrary numerical vector into an information vector.

The operation of the recognition algorithm can be schematically represented as follows.

Feature description of an object $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

↓ Recognition algorithm B

Vector of estimates for a class $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$

↓ Decision rule C

Information vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_l)$.

Thus, during the solution of a recognition problem, the object of recognition, i.e., an image, is described by three different vectors: the n-dimensional vector of features, the l-dimensional vector of estimates for a class, and the l-dimensional information vector.

Let us briefly recall the pattern recognition problem in the standard statement that was formulated by Zhuravlev.

$Z(I_0, S_1, \ldots, S_q, P_1, \ldots, P_l)$ is a recognition problem, where I_0 is admissible initial information; $S_1, \ldots, S_q$ is the set of admissible objects described by feature vectors; $K_1, \ldots, K_l$ is a set of classes; and $P_1, \ldots, P_l$ is a set of predicates on the admissible objects, $P_i = P_i(S)$, i =1, 2, …, l. Problem Z consists in finding the values of the predicates $P_1, \ldots, P_l$.

Definition. An algorithm is said to be correct for problem Z if the following equality holds: $A\left(I, S_1, \ldots, S_q, P_1, \ldots, P_l\right) = \left\|\alpha_{ij}\right\|_{q \times l}$, where $\alpha_{ij} = P_j(S_i)$.

One of the main tasks of pattern recognition is searching for an algorithm that correctly solves the image recognition problem. Zhuravlev proves the existence theorem for such an algorithm stating that the algebraic closure of AECs for the image recognition problem is correct. AECs are based on the formalization of the concepts of precedence or partial precedence: an algorithm analyzes the proximity between the parts of descriptions of earlier classified objects and the object to be recognized.

Suppose we are given standard descriptions of objects $\{\tilde{S}\}, \tilde{S} \in K_j$ and $\{S'\}, S' \notin K_j$, and a method for determining the degree of proximity between certain parts of the description of *S* and the corresponding parts of the descriptions $\{I(\tilde{S})\}, \{I(S')\}$; *S, j=1,2,...,l,* is the object of recognition. Calculating estimates for the proximity between the parts of the descriptions $\{I(\tilde{S})\}$ and $\{I(S')\}$ and, respectively, between $I(S)$ and $I(S')$, one can construct a generalized estimate for the proximity between S and the sets of objects $\{\tilde{S}\}, \{S'\}$ (in the simplest case, the generalized estimate is equal to the sum of estimates for the proximity between the parts of descriptions). Then, using the set of estimates, one forms a general estimate of an object over a class, which is precisely the value of the membership function of the object in the class.

For the algebraic closure of the AECs, the following existence theorem for an AEC is proved, which correctly solves recognition problem Z.

Theorem. Suppose that natural assumptions on the difference between the descriptions of classes and recognition objects hold for the vectors of features in recognition problem Z. Then the algebraic closure of the class of AECs is correct for problem Z.

The image recognition problem is one of the classical examples of problems with incompletely formalized and partially contradictory data. This suggests that the application of an algebraic approach to image recognition may lead to important results; hence, the “algebraization” of this field is the most promising approach for development of the required mathematical apparatus for the analysis and estimation of information represented by images.

When the recognition objects are images, this theorem cannot be directly applied. There are several reasons for this. First, representation of an image by a vector of features (as in the case of a standard recognition object) often leads to the loss of a considerable part of the information about the image and, consequently, to an incorrect classification. Second, the existence of equivalence classes is an essential difference between the image recognition problem and the recognition problem in the classical formulation.

Passage from the algebra of pattern recognition algorithms to an algebra of image recognition algorithms requires a choice, first, of algorithms used as elements of algebra, and second, of algebraic representations of images that make it possible to formalize the task of choosing descriptors. It is expedient to select representations taking into account the possibility of combining the initial information and algorithms of different types. For the first time, the idea of a combination of qualifiers with optimization of their operation by algebraic correction was suggested and justified by Yu.I. Zhuravlev. The complex of mathematical methods related to synthesis and research of such qualifiers is known under the common title “Algebraic Approach to Tasks of Recognition, Classification, and Prediction.” In the English-Language literature for the designation of qualifiers, the term Multiple Classifiers (Winbridge, Kittler, 2001) is used. Recently, quite interesting results have been achieved in the field of theoretical-informational analysis of combined qualifiers (Grin et al., 2001), developments of specific strategies for merging algorithms (Kittler, Alkoot, 2001), and usage of methods of code theory in tomography (Tax, Duin, 2001).

Image analysis and understanding have a certain peculiarity, due to which the use of the Zhuravlev algebraic approach in the general form is inconvenient. The reasons are the following:

- the character of the considered problem is not taken into account if algebraic methods are applied to the information represented in the form of images;
- the results of application of the theory cannot always be simply interpreted;
- there are many natural transformations of images which are easily interpreted from the user’s point of view (for instance, rotation, contraction, stretching, colour inversion, etc.) but are hardly representable by standard algebraic operations.

The necessity arises of using algebraic tools to record natural transformations of images. Moreover, the algebraization of image analysis and understanding must include the construction of algebraic descriptions of both the images themselves and algorithms for their processing, analysis, and recognition.

Analysing the publications related to applications of algebraic methods to image analysis and understanding, we distinguish the following advantages of unified representation of images and algorithms for their processing and analysis:

- construction of unified representations for descriptions of images;
- efficiency of transition from input data in the form of images to different formal models of the images;
- naturalness of uniting the algebraic representation of the information with the developed algebraic tools for pattern recognition, which has been successfully
- employed;
- the possibility of using the methods of mathematical modelling employed in applied domains to which the processed images belong;
- the possibility of using the image descriptions in the form of group-theoretic representations;
- naturalness of uniting the methods of structural analysis of images with tools of probabilistic analysis;
- the possibility of a formalized description for problems of parallelizing with due regard for the specifics of particular computational architectures.

The “algebraic approach” to solve the tasks of classification and/or pattern recognition was developed in the school of Yu. Zhuravlev starting from 1960s as means to build the correct algorithms (i.e. the algorithms that produce zero errors on a control set of objects) over specified sets of features. Within the framework of the algebraic approach, the algorithms are built as compositions of type where A is the entire algorithm, B is an operator “base classifier” that maps the feature space into a matrix of estimates of the assignments of the objects’ classes, C is the “decision rule” operator that maps the matrix of estimates into binary matrix of the answers of the entire algorithm A.

In the framework of scientific school of Yu.I.Zhuravlev several essential results were obtained in algebraic direction by V.L.Matrosov

(Matrosov, 1985), (Khachai, 2010), by K.V.Rudakov (Rudakov, 1987, 1988), (Rudakov, Vorontsov, 1999), (Rudakov, Chekhovich, 2005, 2007) and V.D.Mazurov (Mazurov, 1971), (Mazurov, Khachai, 2007).

Apart from basic researches of Yu.I.Zhuravlev scientific school there are significant number of papers concerned with algebraic methods of analysis and estimation of information represented as signals, in partially V.G.Labunec (Labunec, 1984), Yu.P.Pityev (Pityev, 2004), I.N.Sinicyn (Sinicyn, 2007), Ya.A.Furman (Furman, 2009), (Furman et al., 2012), V.M.Chernov (Chernov, 2001, 2007), (Felsberg et al., 2000).

4 IMAGE ALGEBRA

Mathematical morphology (Serra, 1982), (Soille, 1996, 2003, 2004), (Sternberg, 1980, 1985, 1986), proposed by Minkowski and Hadwiger and developed by Matheron and Serra, seems to be the first attempt to create a theoretical apparatus that allows one to describe many widespread operations of image processing in the composition of a rather small set of standard simple local operations. Such representations allow one to formalize the choice of procedures for image processing and are convenient for implementation on parallel architectures. It might have been the success of mathematical morphology that initiated numerous attempts of algebraization both in the domain of algorithm representations and in closed domains. Mathematical morphology is an efficient tool for uniform representation of local operations of image processing, analysis, and understanding in terms of algebras over sets. It makes it possible to describe algorithms for image transformations in terms of four basic local operations, namely, those of erosion, dilatation, opening, and closing; moreover, any two of these operations form a basis, in terms of which the other two operations may easily be expressed. This is very convenient for the development of software systems, in which the user can quickly design particular algorithms from basic blocks.

On the basis of mathematical morphology, Sternberg (Sternberg, 1980, 1985, 1986) introduced the concept of an image algebra.

The image algebra made it possible to represent algorithms for image processing in the form of algebraic expressions, where variables are images and operations are geometrical and logical transformations of the images. It is known that the possibilities of mathematical morphology are very limited. In particular, many important and widely used operations of image processing (feature extraction based on the convolution operation, Fourier transforms, use of the chain code, equalization of a histogram, rotations, recording, and nose elimination), except for the simplest cases, can hardly (if ever) be realized in the class of morphological operations.

The impossibility of constructing a universal algebra for tasks of image processing on the basis of the morphological algebra may be explained by the limitation of the basis consisting of the set-theoretical operations of addition and subtraction in Minkowski's sense.

It is known that this basis has the following drawbacks (Miller, 1983): complicated realization of widely used operations of image processing; impossibility of establishing a correspondence between the operations of mathematical morphology and linear algebra; impossibility of using mathematical morphology for transformations between different algebraic structures, in particular, sets including real and complex numbers and vector quantities.

These problems have been solved in the standard image algebra (IA) by G. Ritter (Ritter, 2001), (Ritter, Wilson, 2001) on the basis of a more general algebraic representation (Birkhoff, Lipson, 1970) of operations of image processing and analysis. Standard Image Algebra by G.Ritter is a unified algebraic representation of image processing and analysis operations. Image algebra generalizes the known local methods for image analysis, in particular, mathematical morphology, and provides the following advantages as compared with mathematical morphology: it makes it possible to work with both real and complex quantities; it allows one to include both scalar and vector data into the input information; it makes image-algebra structures consistent with linear structures; it provides a more accurate and complete description of its operations and operands; with the help of a special structure "template," composite operations of image processing are divided into a number of parallel simplest operations.

The bottleneck in applications of methods of image algebra to image recognition is the choice of the sequence of algebraic operations and templates for representation of composite operations of image processing.

At present, this choice is based, as a rule, on general representations of the character of images and tasks. Deficiencies of this approach are obvious: first, it is subjective and its success depends to a great extent on the user's experience and, second, it is intended to solve a specific narrow class of problems. Image algebra generalizes the known local methods for image analysis, in particular, mathematical morphology.

Investigations in the area of algebraization and image analysis of the 1970–1980s represent a source of development of the descriptive image algebra (DIA). DIA by I.Gurevich is a unified algebraic language for describing, performance estimating and standardizing representation of algorithms for image analysis, recognition and understanding as well as image models.

An object that lies most closely to the developed mathematical object is the image algebra proposed and developed by Ritter (Ritter, 2001). Ritter's main goal in developing the image algebra is the design of a standardized language for description of algorithms for image processing intended for parallel execution of operations. A key difference in the new image algebra from the standard Ritter image algebra is that DIA is developed as a descriptive tool, i.e., as a language for description of algorithms and images rather than a language for algorithm parallelizing.

The conceptual difference of the algebra under development from the standard image algebra is that objects of this algebra are (along with algorithms) descriptions of input information. DIA generalizes the standard image algebra and allows one to use (as ring elements) basic models of images and operations on images or the models and operations simultaneously. In the general case, a DIA is the direct sum of rings whose elements may be images, image models, operations on images, and morphisms. As operations, we may use both standard algebraic operations and specialized operations of image processing and transformations represented in an algebraic form. To use DIA actively, it is necessary to investigate its possibilities and to attempt to unite all possible algebraic approaches, for instance, to use the standard image algebra as a convenient tool for recording certain algorithms for image processing and understanding or to use Grenander's concepts for representation of input information.

In the 1980s, Sternberg formalized the notion "image algebra" and introduced the following definition.

Definition 1. Image algebra is the representation of algorithms for image processing on a cellular computer in the form of algebraic expressions whose variables are images and whose operations are procedures for constructing logical and geometrical combinations of images.

This image algebra is described on the basis of mathematical morphology and is identified by the author with mathematical morphology. In 1985, Sternberg (Sternberg, 1985) noted that the languages for image processing were being developed for each processor architecture and none of them has been created for one computer and run on another. Ritter's image algebra (Ritter, Wilson, 2001) generalizes mathematical morphology, unites the apparatus of local methods for image analysis with linear algebra, and generates more complex structures. The structure of the standard image algebra may be extended by introducing new operations. Hence, it may be successfully applied in the cases where a satisfactory result cannot be obtained with the help of morphology and linear algebra.

Definition 2. A standard image algebra is a heterogeneous (or multivalued) algebra with a complex structure of operands and operations if the basic operands are images (sets of points) and values and characteristics related to these images (sets of values related to these points).

Analysing the existing algebraic apparatus, we came to the statement of the following requirements on the language designed for recording algorithms for solving problems of image processing and understanding: the new algebra must make possible processing of images as objects of analysis and recognition; the new algebra must make possible operations on image models, i.e., arbitrary formal representations of images, which are objects and, sometimes, a result of analysis and recognition; introduction of image models is a step in the formalization of the initial data of the algorithms; the new algebra must make possible operations on main models of procedures for image transformations; it is convenient to use the procedures for image modifications both as operations of the new algebra and as its operands for construction of compositions of basic models of procedures.

Definition 3. An algebra is called a descriptive image algebra if its operands are either image models (for instance, as a model, we may take the image itself or a collection of values and characteristics related to the image) or operations on images, or models and operations simultaneously.

It should be noted that, due to the variety of "algebras", we should indicate which algebra is meant in definition of DIA. For the generality of the results and extension of the domain of applications of the new algebra, to define DIA with one ring, we use the definition of the classical algebra of Van der Waerden (Waerden, 1971).

Thus, a DIA with one ring must satisfy the properties of classical algebras. A DIA with one ring is a basic DIA, because it contains a ring of elements of the same nature, i.e., either a ring of image models or a ring of operations on images.

To design efficient algorithmic schemes for image analysis and understanding, it is necessary to investigate different types of operands and different types of operations applicable to the chosen operands, which generate the DIA.

5 DESCRIPTIVE APPROACH TO IMAGE ANALYSIS

It was largely the necessity to solve complex recognition problems and develop structural recognition methods and specialized image languages that generated the interest in formal descriptions–models of initial data and formalization of descriptions of procedures of their transformation in the area of pattern recognition (and especially in image recognition in 1960s).

As for the substantial achievements in this "descriptive" line of study, we mention publications by A. Rosenfeld (Rosenfeld, 1979), T. Evans (Evans, 1967, 1969), R. Narasimhan (Narasimhan, 1966, 1967, 1968), R. Kirsh (Kirsh, 1964), A. Shaw (Shaw, 1967, 1968), H. Barrow, A. Ambler, and R. Burstall (Barrow, et al., 1972), S. Kaneff (Kaneff, 1972), K.S.Fu (1972), Schlesinger (Schlesinger, Hlavac, 2002). In 1970s, Yu. I. Zhuravlev proposed the so called "Algebraic Approach to Recognition and Classification Problems" (Zhuravlev, 1998), where he defined formalization methods for describing heuristic algorithms of pattern recognition and proposed the universal structure of recognition algorithms. In the same years, U. Grenander stated his "Pattern Theory" (Grenander, 1976, 1978, 1981, 1993, 1996), where he considered methods of data representation and transformation in recognition problems in terms of regular combinatorial structures, leveraging algebraic and probabilistic apparatus. Both approaches dealt with the recognition problem in its classical statement and did not touch upon representation of initial data in the form of images.

Then, up to the middle of 1990s, there was a slight drop in the interest in descriptive and algebraic aspects in pattern recognition and image analysis.

The main intention of DA is to structure different techniques, operations and representations being applied in image analysis and recognition. The axiomatics and formal constructions of DA establish conceptual and mathematical base for representing and describing images and its analysis and estimation. The DA provides a methodology and a theoretical base for solving the problems connected with the development of formal descriptions for an image as a recognition object as well as the synthesis of transformation procedures for an image recognition and understanding. The analysis of the problems is based on the investigation of inner structure and content of an image as a result of the procedures "constructing" it from its primitives, objects, descriptors, features and tokens, and relations between them.

This section contains a brief description of the principal features of the DA needed to understand the meaning of the introduction of the conceptual apparatus and schemes of synthesis of image models proposed to formalize and systematize the methods and forms of representation of images.

The automation of information extraction from images requires complex use all the features of the mathematical apparatus used or potentially suitable for use in determining transformations of information provided in the form of images, namely in problems of processing, analysis, recognition, and understanding of images.

Experience in the development of the mathematical theory of image analysis and its use to solve applied problems shows that, when working with images, it is necessary to solve problems that arise in connection with the three basic issues of image analysis, i.e., (1) the description (modelling) of images; (2) the development, exploration, and optimization of the selection of mathematical methods and tools for information processing in the analysis of images; and (3) the hardware and software implementation of the mathematical methods of image analysis.

The DA is proposed and developed as a conceptual and logical basis of the extraction of information from images. This includes the following basic tools of analysis and recognition of images: a set of methods of analysis and recognition of images, reducing images to a form suitable for recognition (RIFR) techniques, conceptual system of analysis and recognition image, DIM classes, the DIA language, statement of problems of analysis and recognition of images, and the basic model of image recognition.

The main areas of research within the DA are (1) the creation of axiomatics of analysis and recognition of images, (2) the development and implementation of a common language to describe the processes of analysis and recognition of images (the study of DIA), and (3) the introduction of formal systems based on some regular structures to determine the processes of analysis and recognition of images (see (Gurevich, 1989, 1991)).

Mathematical foundations of the DA are as follows: (1) the algebraization of the extraction of information from images, (2) the specialization of the Zhuravlev algebra (Zhuravlev, 1998) to the case of representation of recognition source data in the form of images, (3) a standard language for describing the procedures of the analysis and recognition of images (DIA) (Gurevich, Yashina, 2006), (4) the mathematical formulation of the problem of image recognition, (5) mathematical theories of image analysis and pattern recognition, and (6) a model of the process for solving a standard problem of image recognition. The main objects and means of the DA are as follows: (1) images; (2) a universal language (DIA); (3) two types of descriptive models, i.e., (a) an image model and (b) a model for solving procedures of problems of image recognition and their implementation; (4) descriptive algebraic schemes of image representation (DASIR); and (5) multimodel and multiaspect representations of images, which are based on generating descriptive trees (GDT).

The basic methodological principles of the DA are as follows: (1) the algebraization of the image analysis, (2) the standardization of the representation of problems of analysis and recognition of images, (3) the conceptualization and formalization of phases through which the image passes during transformation while the recognition problem is solved, (4) the classification and specification of admissible models of images (DIM), (5) RIFR, (6) the use of the standard algebraic language of DIA for describing models of images and procedures for their construction and transformation, (7) the combination of algorithms in the multialgorithmic schemes, (8) the use of multimodel and multiaspect representations of images, (9) the construction and use of a basic model of the solution process for the standard problem of image recognition, and (10) the definition and use of nonclassical mathematical theory for the recognition of new formulations of problems of analysing and recognizing images.

Note that the construction and use of mathematical and simulation models of studied objects and procedures used for their transformation is the accepted method of standardization in the applied mathematics and computer science.

A more detailed description of methods and tools of the DA obtained in the development of its results can be found in (Gurevich, 2005), (Gurevich, Yashina, 2008, 2012).

6 CONCLUSIONS

Practical application of the algebraic instruments DA was demonstrated: we have shown how to build, by means of DIA, the model of a technology for automating diagnostic analysis of cytological preparations of patients with tumors of the lymphatic system. This model has been used for the creation of software for application of this technology, its testing, and comparison of results.

The main contribution is construction of a model for a method ensuring a unified representation of the technology, instead of development of a method for solving a medical task. This work, thus, solves a dual task: first, it represents a technology in the form of a well-structured mathematical model and, second, shows how DIA can be used in an image analysis task.

In the future, DA and its main instruments—DIA, DIM and GDT—will be applied to constructing models of an information technology for automation of diagnostic analysis of medical images in other areas of medicine.

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REFERENCES

Barrow, H.G., Ambler, A.P., Burstall, R.M., 1972. Some Techniques for Recognizing Structures in Pictures. In *Frontiers of Pattern Recognition (The Proceedings of* *the International Conference on Frontiers of Pattern Recognition, ed. Satosi Watanabe), pp.1-30.* Academic Press, New York, London.

Birkhoff, G., Lipson, J.D. Heterogeneous Algebras. In *Journal of Combinatorial Theory, Vol.8, pp. 115-133*.

Chernov, V.M., 2007. On defining equations for the elements of associative and commutative algebras. In *Space-Time Structure. Algebra and Geometry, pp.182-188*. Lilia Print.

Duff, M.J.B., Watson, D.M., Fountain, T.J., Shaw, G.K., 1973. A cellular logic array for image processing. In *Pattern Recognition, vol.5, no.3, pp. 229–247*.

Evans, T.G., 1967. A Formalism for the Description of Complex Objects and 1st Implementation. In *Proceedings of the Firth International Conference on Cybernetics, Namur, Belgium.*

Evans, T.G., 1969. Descriptive Pattern Analysis Techniques: Potentialities and Problems. In *Methodologies of Pattern Recognition (The Proceedings of the International Conference on Methodologies of Pattern Recognition)*, *pp.149-157*. Academic Press, New York, London.

Felsberg, M., Bulov, Th., Sommer, G., Chernov, V.M., 2000. Fast Algorithms of Hypercomplex Fourier Transforms. In *Geometric Computing with Clifford Algebras, pp.231-254*. Springer Verlag.

Fu, K.S., 1972. On syntactic pattern recognition and stochastic languages. In *Frontiers of Pattern Recognition (S.Watanabe, ed.)*. Academic Press, New York.

Furman, Ya. A., 2009. Parallel Recognition of Different Classes of Patterns. In *Pattern Recognition and Image Analysis, Vol.19, No.3, pp.380-393.* Pleiades Publishing, Ltd.

Furman, Ya. A., Eruslanov, R.V., Egoshina, I.L., 2012. Recognition of Images and Recognition of Polyhedral Objects. In *Pattern Recognition and Image Analysis, vol.22, no.1, pp.196-209*. Pleiades Publishing, Ltd.

Grevander, U., 1976, 1978, 1981. Lectures in Pattern Theory. N.Y.: Sprinder-Verlag, V.1; V.2; V.3.

Grenander, U., 1993. *General Pattern Theory*, A Mathematical Study of Regular Structure. Clarendon Press, Oxford.

Grenander, U., 1996. *Elements of Pattern Theory*, The Johns Hopkins University Press.

Grin, J., Kittler, J., Pudil, P., Somol, P., 2001. Information Analysis of Multiple Classifier Fusion. In *Multiple Classifier Systems. Second International Workshop, MCS 2001, Cambridge, UK, pp.168-177*. Springer - Verlag.

Gurevich, I., 1989. The Descriptive Framework for an Image Recognition Problem. In *Proceedings of the 6th Scandinavian Conference on Image Analysis, vol. 1, pp. 220 – 227*. Pattern Recognition Society of Finland,

Gurevich, I., 1991. Descriptive Technique for Image Description, Representation and Recognition. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications in the USSR, vol. 1, pp. 50 – 53*. MAIK "Interpreodika".

Gurevich, I., 2005. The Descriptive Approach to Image Analysis. Current State and Prospects. In *Proceedings of 14th Scandinavian Conference on Image Analysis, LNCS 3540, pp. 214-223*. Springer-Verlag Berlin Heidelberg.

Gurevich, I., Jernova, I., 2003. The Joint Use of Image Equivalents and Image Invariants in Image Recognition. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications*, *vol. 13, no.4, pp. 570-578*. MAIK "Nauka/Interperiodica".

Gurevich, I., Koryabkina, I., 2006. Comparative Analysis and Classification of Features for Image Models. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications, vol.16, no.3, pp. 265-297.* MAIK "Nauka/Interperiodica"/Pleiades Publishing, Inc.

Gurevich, I., Yashina, V., 2006. Operations of Descriptive Image Algebras with One Ring. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications, vol.16, no.3, pp. 298-328*. MAIK "Nauka/Interperiodica"/Pleiades Publishing, Inc.

Gurevich, I., Yashina, V., 2006. Computer-Aided Image Analysis Based on the Concepts of Invariance and Equivalence. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications, vol.16, no.4, pp.564-589*. MAIK "Nauka/Interperiodica"/Pleiades Publishing, Inc.

Gurevich, I., Yashina, V., 2008. Descriptive Approach to Image Analysis: Image Models. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications*. MAIK "Nauka/Interperiodica"/Pleiades Publishing, Inc., vol.18, no.4, pp. 518-541.

Gurevich, I., Yashina, V., Koryabkina, I., Niemann, H., Salvetti, O., 2008. Descriptive Approach to Medical Image Mining. An Algorithmic Scheme for Analysis of Cytological Specimens. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications, vol.18, no.4, pp. 542-562*. MAIK "Nauka/Interperiodica"/Pleiades Publishing, Inc.

Gurevich, I., Yashina, V., 2012. Descriptive Approach to Image Analysis: Image Formalization Space. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications*, v*ol. 22, no. 4, pp. 495-518.* Pleiades Publishing, Inc.

Haralick, R.M., Sternberg, S.R., Zhuang, X. Image Analysis Using Mathematical Morphology. In *IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. PAMI-9, No. 4, pp.532-550.*

Kaneff, S., 1972. Pattern Cognition and the Organization of Information. In *Frontiers of Pattern Recognition (The Proceedings of the International Conference on Frontiers of Pattern Recognition, ed. Satosi Watanabe), pp.193-222*. Academic Press, New York, London.

Khachai, M.Yu., 2010. Computational complexity of recognition learning procedures in the class of piecewise-linear committee decision rules. In

Automation and Remote Control, vol. 71, no. 3, pp. 528-539.

Kirsh, R., 1964. Computer Interpretation of English Text and Picture Patterns. In *IEEE-TEC, Vol. EC-13, No. 4.*

Kittler, J., Alkoot, F.M., 2001. Relationship of Sum and Vote Fusion Strategies. In *Multiple Classifier Systems. Second International Workshop, MCS 2001, Cambridge, UK, pp.339-348*. Springer - Verlag.

Labunec, V.G., 1984. *Algebraic theory of signals and systems (digital signal processing)*. Krasnoyarsk University.

Malcev, A. I., 1973. *Algebraic Systems*. Springer-Verlag, Berlin.

Matrosov, V.L., 1985. The capacity of polynomial expansions of a set of algorithms for calculating estimates. In *USSR, Comput.Maths.Math.Phys., Printed in Great Britain, Vol.25, No.1, pp.79-87*.

Mazurov, V.D., 1971. Committees of inequalities systems and the recognition problem. In *Cybernetics,* vol. 7, no. 3, pp. 559-567.

Mazurov, V.D., Khachai, M.Yu., 2007. Parallel computations and committee constructions. In *Automation and Remote Control*, *vol. 68, no. 5, pp. 912-921.*

Narasimhan, R., 1966. Syntax-Directed Interpretation of Classes of Pictures. In *Community ACM, vol.9, no.3.*

Narasimhan, R., 1967. Labeling Schemata and Syntactic Descriptions of Pictures. In *Information and Control, vol. 7, no. 2*.

Narasimhan, R., 1968. On the Description, Generalization and Recognition of Classes of Pictures. In *NATO Summer School on Automatic Interpretation and Classification of Images, Pisa, Italy*.

Narasimhan, R., 1970. Picture Languages. In *Picture Language Machines (ed. S.Kaneff), pp.1-30*. Academic Press, London, New York.

Pavel, M., 1976. Pattern Recognition Categories. In *Pattern Recognition*, *Vol.8, No.3.pp. 115-118.*

Pavel, M., 1989. *Fundamentals of Pattern Recognition*. New York, Marcell, Dekker, Inc..

Pytiev, Yu.P., 2004. *Method of mathematical modeling of measuring and computing systems*. MAIK Nauka, Moscow (sec.ed) [in Russian].

Ritter, G.X., Sussner, P., Diaz-de-Leon, J.L., 1998. Morphological associative memories. In *IEEE Trans. on Neural Networks, Vol. 9, No. 2, pp. 281–292.*

Ritter, G.X., Sussner, P., 1996. Introduction to Morphological Neural Networks. In *Proceedings of ICPR 1996, IEEE, pp. 709-716*.

Ritter, G.X., Diaz-de-Leon, J.L., Sussner, P., 1999. Morphological bidirectional associative memories. In *Neural Networks, vol. 12, pp. 851–867.*

Ritter, G., Wilson, J., 2001. *Handbook of Computer Vision Algorithms in Image Algebra*. 2-d Edition. CRC Press Inc..

Ritter, G., 2001. *Image Algebra*. Center for computer vision and visualization, Department of Computer and Information science and Engineering, University of Florida, Gainesville, FL 32611.

Ritter, G.X., Gader, P.D., 1987. Image Algebra techniques for parallel image processing. In *Parallel Distributed Computers, Vol.4, no.5, pp.7-44*.

Ritter, G.X., Wilson, J.N. , Davidson, J.L., 1990. Image Algebra: An Overview. In *Computer Vision, Graphics, and Image Processing, vol.49, pp.297-331*.

Rudakov, K. V., 1987. Universal and local constraints in the problem of correction of heuristic algorithms. In *Cybernetics*, *vol. 23, no. 2, pp. 181-186.*

Rudakov, K. V., 1987. Completeness and universal constraints in the correction problem for heuristic classification algorithms. In *Cybernetics, vol. 23, no. 3, pp 414-418*.

Rudakov, K. V., 1988. Application of universal constraints in the analysis of classification algorithms. In *Cybernetics, vol. 24, no. 1, pp 1-6*.

Rudakov, K. V. , Vorontsov, K. V., 1999. Methods of Optimization and Monotone Correction in the Algebraic Approach to the Recognition Problem. In *Dokl. Akad. Nauk 367, 314–317 [Dokl. Math. 60, 139–142]*.

Rudakov, K. V., Chekhovich, Yu. V., 2005. Completeness Criteria for Classification Problems with Set-Theoretic Constraints. In *Computational Mathematics and Mathematical Physics, vol. 45, no. 2, pp. 329-337.*

Serra, J., 1982. *Image Analysis and Mathematical Morphology.* London, Academic Press.

Sinicyn, I.N., 2007. *Calman and Pugachev Filters*. Logos, Moscow (sec.ed.) [in Russian].

Shaw, A. A Proposed Language for the Formal Description of Pictures. In *CGS Memo. 28, Stanford University*.

Shaw, A, 1967. The Formal Description and Parsing of Pictures". *Ph.D. Thesis, Computer Sciences Department, Stanford University, December 1967 (also Tech. Rept CS94, April 1968)*.

Schlesinger, M., Hlavac, V., 2002. Ten Lectures on Statistical and Structural Pattern Recognition. In *Computational Imaging and Vision*, vol. 24, p.520. Kluwer Academic Publishers - Dordrecht / Boston / London.

Soille, P., 1996. Morphological partitioning of multispectral images. In *Journal of Electronic Imaging, vol.5, no.3, pp. 252-265*.

Soille, P., 2003, 2004. *Morphological Image Analysis. Principles and Applications (Second Edition).* Springer-Verlag Berlin Heidelberg, New York.

Sternberg, S.R., 1980. Language and Architecture for Parallel Image Processing. In *Proceedings of the Conference on Pattern Recognition in Practice*. Amsterdam.

Sternberg, S., 1985. *An overview of Image Algebra and Related Architectures*. Integrated Technology for parallel Image Processing (S. Levialdi, ed.), London: Academic Press.

Sternberg, S.R., 1986. Grayscale morphology. In *Computer Vision, Graphics and Image Processing, vol.35, no.3, pp. 333-355*.

Tax, D.M.J., Duin, R.P.W., 2001. Combining One-Class Classifiers". Multiple Classifier Systems. In *Second*

International Workshop, MCS 2001, Cambridge, UK. Springer - Verlag.

Unger, S.H., 1958. A computer oriented toward spatial problems. In *Proceedings of the IRE, vol.46, pp. 1744-1750.*

Van Der Waerden, B.L., 1971. *Algebra I, Algebra II*, Springer-Verlag, Berlin Heidelberg New York.

Winbridge, D., Kittler, J., 2001. Classifier Combination as a Tomographic Process. In *Multiple Classifier Systems. Second International Workshop, Cambridge, UK, pp. 248 – 258*. Springer - Verlag.

Zhuravlev, Yu., 1998. An Algebraic Approach to Recognition and Classification Problems. In *Pattern Recognition and Image Analysis: Advances in Mathematical Theory and Applications, vol.8, pp.59-100*. MAIK "Nauka/Interperiodica".