Spatiotemporal Complex Geometrical Optics (CGO) of N 3D Interacting Asymmetric Gaussian Wave Packets in Nonlinear Medium

CGO as the Simplest and Efficient Method for Spatiotemporal Evolution

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Abstract: The complex geometrical optics (CGO) was applied for the spatiotemporal evolution of arbitrary number of 3D mutually incoherent (with different carrier frequencies) Gaussian wave packets (GWPs) interacting and propagating in a nonlinear medium of Kerr type. The CGO reduced description of the propagation of the beam, the pulse and the wave packet to complex ordinary differential equations (ODE) This leads to exceptionally fast numerical algorithms. We observed high efficiency of the CGO method to compute interactions of arbitrary number of 3D Gaussian wave packets propagating in a nonlinear (anomalous) dispersive medium of the Kerr type. The derived CGO equations were compared with equations obtained by the variational method. CGO described the Gaussian beam propagation in free space as well as the Gaussian pulse spreading in the linear anomalous dispersive medium more illustratively than both the Fourier transform method and the Fresnel diffraction integral method. The spatiotemporal CGO has been proven to be a method more practical than the spectral analysis, the variational method, the method of moments and the method of the generalized eikonal approximation. Complementary to the presented results, an on-line CGO solver, implemented in Javascript, is freely available at the authors' website: http://slawek.ps.pl/odelia.html.

1 INTRODUCTION

In traditional understanding, the geometrical optics (GO) is the method to create pictures based on geometrical evolutions of light rays. However, in a wider recognition GO can be classified to be an approximated method to describe the wave motion of wave fields such as beams, pulses and wave packets. Such wider recognition was first established by papers and books of Debye, Sommerfeld, Runge and Luneburg (Sommerfeld, 1964; Luneburg, 1964; Kline and Kay, 1965; Born and Wolf, 1959), who proposed the geometrical form of the wave field representation and who derived from the wave equation two geometrical optics equations: the transport equation describing the amplitude and the eikonal equations, which determine the phase of the signal. Operation of the first laser in 1960 emphasized the importance of concepts of Sommerfeld, Runge, Born, Wolf and Luneburg to describe the evolution of the laser fundamental mode in the form of Gaussian beam (GB). At the beginning of the 1960's, Kravtsov proposed the method of geometrical optics based on the concepts of complex rays (Kravtsov, 1967), the bundle of which can represent the Gaussian wave field (Deschamps, 1971) and which enables to include the effect of a spatially limited beam diffraction in the free space into the scope of the classical geometrical ray description. Contemporaneously, Kogelnik (Kogelnik, 1965) proposed the representation of a collimated beam by introduction of the quasi-optical complex parameter 1/q, which describes the evolution of two beam parameters: the GB width and the radius of the curvature in the one single complex quantity, which can be transformed through lens-like media, including resonators and even more advanced optical systems. Nowadays, the Kogelnik transformation laws are expressed in a more general convenient form

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referred to as the ABCD matrix method, or the the generalized beam matrix method (for instance generalized to ABCDGH matrix). Another quasi-optical method is the nonlinear geometrical optics (NGO) proposed by (Akhmanov, Sukhorukov and Khokhlov, 1968), which uses the parabolic equation to describe self-focusing of a GB in a nonlinear medium of Kerr type. Nowadays, the NGO formalism is often recognized as a generalized eikonal approximation (Yap, Quek and Low, 1998), which was used at the end of the 1990's to describe the propagation of stationary electromagnetic waves in linear and nonlinear media and the evolution of electromagnetic pulse in a linear dispersive medium. During the 1960's/70's, at least four other methods were proposed to describe optical beams: the dynamic ray tracing (Luneburg, 1964; Deschamps and Mast, 1964; Arnaud 1976), which describes geometrical spreading of the rays using Hamilton equations; the method of inhomogeneous wave tracking (Choudhary and Felsen, 1974; Felsen, 1976), which bases on some specific regularities concerning evolution of the family of phase paths and wave fronts. Finally, evolution of the phase paths and wave fronts enables to construct the wave field of the beam; the method of moments (MM) (Vlasov, Petrischev and Talanov, 1971), which deals with intensity moments satisfying the parabolic equation and allowing to determine the power of the wave beam, the centre of the beam intensity, the beam divergence and the evolution of the effective beam radius; the variational method (Anderson, 1983), called also the Ritz method, which bases on semi-analytical approach to the pulse and the beam propagation by means of the standard variational principle. In 2004, the CGO method dealing with the complex eikonal and the complex amplitude was proposed to describe the first GB diffraction in homogeneous and inhomogeneous media (Berczynski and Kravtsov, 2004). The CGO method enables to reduce immediately the complicated spatial and temporal description based on partial differential equations to solving ordinary differential equations contrary to other methods detailed above. The CGO method was next applied to describe the self-focusing of GBs in nonlinear inhomogeneous fibres (Berczynski, 2011) and in nonlinear saturable media with absorption (Berczynski, 2013). First of all, we demonstrate in this paper that CGO method enables to perform fast and effective numerical calculations based on ODEs. In this way, the complexity and problems related to nonlinear optics can be solved and analysed in more illustrative and transparent way.

2 FIRST-ORDER COMPLEX EQUATIONS OF CGO METHOD

CGO method deals with eikonal equation, which, for spatiotemporal evolution of a 3D wave packet in a medium with relative permittivity ε and anomalous dispersion, can be presented in a convenient form

$$\left(\frac{\partial\psi'}{\partial\eta_1'}\right)^2+\left(\frac{\partial\psi'}{\partial\eta_2'}\right)^2+\left(\frac{\partial\psi'}{\partial z'}\right)^2+\left(\frac{\partial\psi'}{\partial\tau'}\right)^2=\varepsilon \tag{1}$$

In Eq. (1) $\tau'=(t-k_\omega' z)\sqrt{k_0/|k_\omega''|}$ denotes scaled time, $\eta_1'=k_0\eta_1$, $\eta_2'=k_0\eta_2$ are dimensionless transverse coordinates, $z'=k_0 z$ denotes dimensionless propagation direction and $\psi'=k_0\psi$ is dimensionless eikonal, where $k_0=2\pi/\lambda_0$ (λ_0 is the wavelength in vacuum). Within the spatiotemporal CGO method, eikonal ψ' is complex-valued and takes the form

$$\psi'=z'+\sum_{i=1}^{3}B_i(z')\eta_i'^2/2\,,\quad i=1,2,3 \tag{2}$$

$B_i=B_i(z)$ denotes complex functions with introduced convenient notation including temporal coordinate $\eta_3'=\tau'$. The real parts of B_i embrace spatial and temporal chirps κ_i whereas imaginary parts include widths w_i. Thus, the form of the complex parameters $B_i=B_i(z)$ is as follows:

$$B_i=B_{Ri}+iB_{Ii}=\kappa_i+i/w_i^2 \tag{3}$$

Expanding subsequently relative permittivity $\varepsilon=\varepsilon(z',\eta_1',\eta_2',\tau')$ in Eq. (1) in Taylor series in η_1',η_2',τ' and substituting Eq. (3) into Eq. (1) we obtain first-order Riccati type equations in the form

$$\frac{dB_i}{dz'}+B_i^2=\alpha_i \tag{4}$$

$$\alpha_{1,2}=\frac{1}{2}\frac{\partial^2\varepsilon}{\partial\eta_{1,2}'^{\,2}}\Big|_{\eta_1'=0}\Big|_{\eta_2'=0}\Big|_{\tau'=0}\,,\quad \alpha_3=\frac{1}{2}\frac{\partial^2\varepsilon}{\partial\tau'^2}\Big|_{\tau'=0}\Big|_{\eta_1'=0}\Big|_{\eta_2'=0} \tag{5}$$

Equations in Eq. (4) can expressed by real and imaginary parts of B_i in the form

$$\begin{cases}\dfrac{dB_{Ri}}{dz'}+B_{Ri}^2-B_{Ii}^2=\alpha_i\\[2mm]\dfrac{dB_{Ii}}{dz'}+2B_{Ri}B_{Ii}=0\end{cases} \tag{6}$$

The above equations in Eq. (6) lead to the known relations between the packet widths w_i and chirps κ_i together with ordinary differential equation:

$$\kappa_i = \frac{1}{w_i}\frac{dw_i}{dz'}, \quad \frac{d^2 w_i}{dz'^2} - \alpha_i w_i = \frac{1}{w_i^3} \tag{7}$$

Transport equation:

$$div(A^2 \nabla \psi') = 0 \tag{8}$$

for CGO representation of eikonal in Eq. (2) together with paraxial approximation leads immediately to a first-order ordinary differential equation for complex amplitude in the form

$$\frac{dA}{dz'} + \frac{1}{2}\left(\sum_{i=1}^{3} B_i\right) A = 0 \tag{9}$$

By integrating the above equation, we obtain the complex amplitude of Gaussian wave packet in the form

$$A(z') = A_0 \exp\left(-\frac{1}{2}\int\left[\sum_{i=1}^{3} B_i(z)\right]dz'\right) \tag{10}$$

where $A_0 = A(0)$ is the initial amplitude. The modulus of complex amplitude has the form

$$|A(z')| = |A_0| \exp\left(-\frac{1}{2}\int\left[\sum_{i=1}^{3}(B_{Ri}(z))\right]dz'\right) \tag{11}$$

where $B_{Ri} = \mathrm{Re}\, B_i$. The first integral of the second equation in Eq. (6) leads to the dependences

$$B_{Ii} / B_{Ii}(0) = \exp\left(-2\int B_{Ri} dz'\right) \tag{12}$$

Using Eq. (3), we obtain spatiotemporal energy flux conservation principle

$$w_1 w_2 w_3 |A|^2 = w_{01} w_{02} w_{03} |A_0|^2 \tag{13}$$

3 CGO, GEA, FOURIER TRANSFORM, FRESNEL DIFFRACTION INTEGRAL. COMPARATIVE ANALYSIS

For the readers' convenience, let us now compare the efficiency of the CGO method with GEA formalism on the example of a quite fundamental evolution of wave packet in a linear dispersive medium. The two methods mentioned above are based on the analogous starting equations but CGO method which uses complex quantities (eikonal and amplitude) reduces automatically the description based on partial differential equations to solving ordinary differential equations for the packet parameters, which is demonstrated below.

Following (Yap, Quek and Low, 1998), let us define first using GEA approach real valued phase L and real valued envelope ϕ of the optical signal. The fundamental disadvantage of the GEA approach lies in the fact that both the amplitude and the phase depend on all spatial and temporal parameters (η_1, η_2, τ, z), which leads to coupled partial differential equations (generalized eikonal equation and transport equation), which require a lot of efforts to be solved. GEA utilizes a representation of the wave packet in the form

$$u(\mathbf{r}, t) = \phi(\mathbf{r}, \tau)\exp(ik_0 L(\mathbf{r}, \tau)) \tag{14}$$

where $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. Substituting Eq. 14 into the wave equation [Born and Wolf, 1959], the set of partial differential equations is obtained in the complicated form

$$(\nabla L)^2 - k_0 k''_\omega \left(\frac{\partial L}{\partial \tau}\right)^2 = \varepsilon + \frac{1}{k_0^2 \phi}\nabla^2 \phi - \frac{1}{k_0^2 \phi} k_0 k''_\omega \frac{\partial^2 \phi}{\partial \tau^2} \tag{15}$$

$$\nabla \cdot (\phi^2 \nabla L) - k_0 k''_\omega \frac{\partial \phi^2}{\partial \tau}\frac{\partial L}{\partial \tau} - k_0 k''_\omega \frac{\partial^2 L}{\partial \tau^2} = 0 \tag{16}$$

It is notable that the above equations are always coupled even in the simplest case of a homogeneous medium, where relative permittivity ε is constant. Finally, in the paper (Yap, Quek and Low, 1998) we notice the need to apply difficult mathematical formalism of Hamilton optics to solve the above equations to obtain the well known formulas for GB propagation in free space and Gaussian pulse evolution in linear dispersive medium. Let us now explain, why the CGO method has an advantage over GEA and enables essential simplification of the starting equations (eikonal equation in standard form and transport equation) to solve quickly ordinary differential equations. First of all, this CGO simplifications is possible only when we introduce complex amplitude A and complex eikonal ψ instead of real envelope ϕ and real phase L used within GEA approach. It can be noticed that complex form of any parameter, which includes "space" for both real and imaginary parts let us "pack" more information in formally the same quantity. Moreover, this obvious conclusion enables us in some sense to control the dependence of pair of complex functions on spatial and temporal parameters z, η_1, η_2, τ. As it is presented below, this complex generalization enables us to simplify mathematical description substantially. Furthermore, this complex generalization leads to the same results as obtained by GEA. It is noteworthy that Eq. (14) describing a 3D Gaussian wave packet can be presented in the form

$$u = \phi_0 \exp\left(-\sum_{i=1}^{3} \eta_i'^2 / 2w_i^2(z) \right) \cdot \exp\left(ik_0 \left[z + \sum_{i=1}^{3} \kappa_i(z)\eta_i'^2 / 2 + \varphi(z)/k_0 \right] \right) \quad (17)$$

$$\phi(\mathbf{r},\tau) = \phi_0 \exp\left(-\sum_{i=1}^{3} \eta_i'^2 / 2w_i^2(z) \right) \quad (18)$$

$$L(\mathbf{r},\tau) = z + \sum_{i=1}^{3} \kappa_i(z)\eta_i'^2 / 2 + \varphi(z)/k_0 \quad (19)$$

Next, let us present the same wave field to separate two functions: one depending only on propagation distance z and the second including all the possible longitudinal and transverse coordinates with time. Thus, we have

$$u = \phi_0 \exp(i\varphi(z)) \cdot \exp\left(ik_0 \left[z + \sum_{i=1}^{3} \kappa_i \eta_i'^2 / 2 + i\sum_{i=1}^{3} \eta_i'^2 / 2w_i^2 \right] \right) \quad (20)$$

This way, Eq. (20) let us define complex amplitude, which depends only on propagation distance z

$$A = A(z) = \phi_0 \exp(i\varphi(z)) \quad (21)$$

and complex eikonal

$$\psi = \psi(z,\tau,\eta_1,\eta_2) = = z + \sum_{i=1}^{3} \kappa_i \eta_i'^2 / 2 + i\sum_{i=1}^{3} \eta_i'^2 / 2w_i^2 \quad (22)$$

which depends on all possible spatial and temporal parameters . The dependence in the form $A = A(z)$ can be justified by geometrical optics assumption in which the wave packet should be localized (paraxial) in the vicinity of propagation direction z. But, complex eikonal $\psi = \psi(z,\tau,\eta_1,\eta_2)$ in Eq. (22) receives more general interpretation, where the constant real part of eikonal $z + \sum_{i=1}^{3} \kappa_i \eta_i'^2 / 2$ describes evolution of wave fronts in space and time, whereas imaginary part $\sum_{i=1}^{3} \eta_i'^2 / 2w_i^2$ represents evolution of phase-paths described by spatiotemporal rays normal to wave fronts, determining finally the power flow direction in space and time. Thus, CGO representation of spatiotemporal wave object has the form

$$u(\mathbf{r},\tau) = A(z)\exp(ik_0\psi(\mathbf{r},\tau)) \quad (23)$$

When we substitute CGO representation in Eq. (23) into wave equation [Born and Wolf, 1959] we obtain a set of two equations of geometrical optics:

$$(\nabla\psi)^2 = \varepsilon\,, \quad div(A^2\nabla\psi) = 0 \quad (24)$$

which are never coupled in a linear medium. In Eq. (24) the differential operator ∇ is 4-vector which lades to spatiotemporal eikonal equation in the form

$$\left(\frac{\partial\psi}{\partial\eta_1}\right)^2 + \left(\frac{\partial\psi}{\partial\eta_2}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - k_0 k''_\omega \left(\frac{\partial\psi}{\partial\tau}\right)^2 = \varepsilon \quad (25)$$

The complex eikonal equation in Eq. (25) for the case of anomalous dispersion of group velocity $k''_\omega < 0$ and after performing scaling procedures (discussed at the beginning of Sec. 1) can be transformed into the eikonal equation in dimensionless form presented in Eq. (1). In this way CGO method uses eikonal equation in the standard form [Born and Wolf, 1959] and does not to require to the generalized eikonal to be defined as opposed to GEA approach. Moreover, it is notable that when the complex amplitude of 3D wave packet depends only on the propagation distance $A = A(z)$ a number of terms containing transverse derivatives $\partial A/\partial\eta_1$, $\partial A/\partial\eta_2$ and $\partial A/\partial\tau$ automatically vanishes in the transport equation. This fact allow us to reduce immediately the partial differential in Eq. (8) to solving the ordinary differential equation in Eq. (9) taking into account paraxial approximation to describe the localized wave packet in CGO language. For comparative analysis, let us apply now the CGO method to describe a 3D wave packet evolution in a linear (anomalous) dispersive medium. Thus, following CGO procedure we reduce immediately the eikonal equation to solving a first-order differential equation, which for the case of a homogeneous medium has the form

$$\frac{dB_i}{dz'} + B_i^2 = 0\,, \quad i = 1,2,3 \quad (26)$$

The solution of Eq. (26) has the form

$$B_i(z') = \frac{B_i(0)}{1 + B_i(0)z'} \quad (27)$$

For the packet with zero initial chirps and initial spatial widths $w_i(0)$, the initial value of complex CGO parameter B_i has the form $B_i(0) = \kappa_i(0) + i/w_i^2(0) = i/w_i^2(0)$. Including this initial condition we obtain solution for $B_i = B_i(z)$ in the form

$$B_i(z') = \frac{i/w_i^2(0)}{1 + iz'/w_i^2(0)} = \frac{i}{w_i^2(0) + iz'} \quad (28)$$

The real and imaginary parts of the above solution are equal to

$$\operatorname{Re} B_i(z') = \frac{z'}{w_i^4(0) + z'^2}, \quad \operatorname{Im} B_i(z') = \frac{w_i^2(0)}{w_i^4(0) + z'^2} \tag{29}$$

As a result, the spatial packet widths and spatial packet chirps turn out to be

$$w_i = w_i(0)\sqrt{1 + \left(\frac{z'}{L'_{Di}}\right)^2}, \quad \kappa_{Wi} = \frac{z'}{z'^2 + L_{Di}^2} \tag{30}$$

where $L'_{Di} = w_i^2(0)$ denotes dimensionless diffraction distance for dimensionless spatial packet widths $w_1(0)$ and $w_2(0)$. We would like to emphasize that the solutions in Eq. (30) are identical with that obtain using Fresnel diffraction integral (Berczynski, Marczynski, 2014). CGO solutions for temporal width and chirp have the form

$$\sigma = \sigma(0)\sqrt{1 + \left(\frac{z'}{L'_{DS}}\right)^2}, \quad \kappa_T = \frac{z'}{z'^2 + L_{DS}^2} \tag{31}$$

where $L'_{DS} = k_0 T^2(0)/|k''_\omega| = \sigma^2(0)$ denotes dimensionless dispersion distance for dimensionless temporal width of the packet $\sigma(0)$. It is worth emphasizing that the solutions in Eq. (31) are also identical with results obtained using Fourier transform for Gaussian pulses, when the initial temporal chirp is zero (Sauter, 1996). To compare solutions obtained by CGO and Fourier transform for Gaussian pulse propagating in linear (anomalous) dispersion medium, let us point that the zero initial chirp expressed by the initial condition $B_i(0) = i/w_i^2(0)$ mentioned above is equivalent to the condition $b_0 = 0$ in the book by (Sauter, 1996). However, solutions in Eq. (9.25) (Sauter, 1996) for Δt (FWHM) and σ_t should be modified a little to be compared with CGO solutions in Eq. (31), taking into account the slightly different definitions of a_0 in Eq. (9.1) in the form $\widetilde{E}(z=0,t) = E_0 \exp(-a_0 t^2)$ as compared with CGO definitions of the width in Eqs. (2,3), where $u = A\exp(-\eta_3^2/2w_3^2) = A\exp(-\tau^2/2\sigma^2)$. Thus, to compare results using the two methods mentioned above, the Eq. (25) presented by (Sauter, 1996) should be converted using substitution: $4a_0^2 = 1/\sigma_0^2$. Summarizing this comparable analysis, it can be stated that the CGO method gives us the same solutions as Fourier transform and Fresnel diffraction integral when we describe a 3D wave packet propagating in a linear dispersive (anomalous) medium. However, it can be observed that replacement of Fourier transform and Fresnel diffraction integral by CGO method means that integrating transform procedures in space and time are substituted by first order complex differential equation shown in Eq. (26), which embraces both spatial and temporal effects. This way, we have proven that the CGO method essentially simplifies the description of wave motion of beams, pulses and wave packets as compared with generalized eikonal approximation method and as compared with classical integral spectral methods, yielding identical solutions.

4 INTERACTION AND EVOLUTION OF ARBITRARY NUMBER OF 3D GAUSSIAN WAVE PACKETS IN NONLINEAR MEDIUM

Let us now generalize CGO method to describe an arbitrary number N of 3D wave packets propagating in a nonlinear medium of Kerr type. Thus, the single Eq. (1) takes a form on N coupled eikonal equations

$$\left(\frac{\partial \psi'_i}{\partial \eta'_1}\right)^2 + \left(\frac{\partial \psi'_i}{\partial \eta'_2}\right)^2 + \left(\frac{\partial \psi'_i}{\partial z'}\right)^2 + \left(\frac{\partial \psi'_i}{\partial \tau'}\right)^2 = \varepsilon(\omega_i, u_1, u_2, \ldots, u_N) \tag{32}$$

with N permittivities depending on each of carrier frequencies of N 3D wave packets in the form

$$\varepsilon(\omega_i, u_1, u_2, \ldots, u_N) = \varepsilon_{i0} + \frac{1}{2}\left(\varepsilon_{i1}|u_1|^2 + \varepsilon_{i2}|u_2|^2 + \ldots + \varepsilon_{iN}|u_N|^2\right) \tag{33}$$

The single transport equation in Eq. (8) takes a form of N equations

$$\frac{\partial \psi'_i}{\partial z'}\frac{d}{dz'}\left(A_i^2\right) + \left[\frac{\partial^2 \psi'}{\partial \eta_1'^2} + \frac{\partial^2 \psi'}{\partial \eta_2'^2} + \frac{\partial^2 \psi'}{\partial z'^2} + \frac{\partial^2 \psi'}{\partial \tau'^2}\right] A_i^2 = 0 \tag{34}$$

leading to spatiotemporal energy flux conservation principles for each of the wave packets in the form

$$w_{i1} w_{i2} w_{i3} |A_i|^2 = w_{i1}(0) w_{i2}(0) w_{i3}(0) |A_i(0)|^2 \tag{35}$$

Applying the CGO procedure in Sect. 1, we obtain next generalized complex Riccati equations in the form

$$\sqrt{\varepsilon_{i0}}\frac{dB_{ij}}{dz'} + B_{ij}^2 = \alpha_{ij}, \quad i = 1, \ldots, N, \quad j = 1,2,3 \tag{36}$$

where

$$\alpha_{ij} = \frac{1}{2}\frac{\partial^2}{\partial \eta_j'^2}\left\{\varepsilon_{i0} + \frac{1}{2}\sum_{l=1}^{N} \varepsilon_{il}|u_l|^2\right\}_{\eta'_j = 0} \tag{37}$$

Subsequently, from Eqs. (36,37) we obtain a set of first order ordinary differential equations:

$$\frac{d}{dz'} w_{ij}(z') = w_{ij}\kappa_{ij}(z') \tag{38}$$

$$\frac{d}{dz'}\kappa_{ij}(z') = -\left\{\kappa_{ij}(z')\right\}^2 + \\ + \frac{1}{\sqrt{\varepsilon_{i0}}}\left\{\left(\frac{1}{w_{ij}(z')}\right)^4 - \frac{1}{2}\sum_{k=1}^{N}\left(\varepsilon_{ik}\left[\frac{|A_k(0)|}{w_{kj}(z')}\right]^2 \prod_{l=1}^{3}\frac{w_{kl}(0)}{w_{kl}(z')}\right)\right\} \tag{39}$$

We would like to emphasize that the above set of Eqs. (38,39) takes a very simple form to be implemented effectively in Matlab or Octave environments. It can be observed that when we limit our consideration to describe a single 3D wave packet propagating in a nonlinear medium of Kerr type, we can also use a variational procedure in the form

$$\iiiint L\left(u,u^*,\frac{\partial u}{\partial z},\frac{\partial u^*}{\partial z},\frac{\partial u}{\partial \eta_1},\frac{\partial u^*}{\partial \eta_1},\frac{\partial u}{\partial \eta_2},\frac{\partial u^*}{\partial \eta_2},\frac{\partial u}{\partial \tau},\frac{\partial u^*}{\partial \tau}\right)dz d\eta_1 d\eta_2 d\tau \quad (40)$$

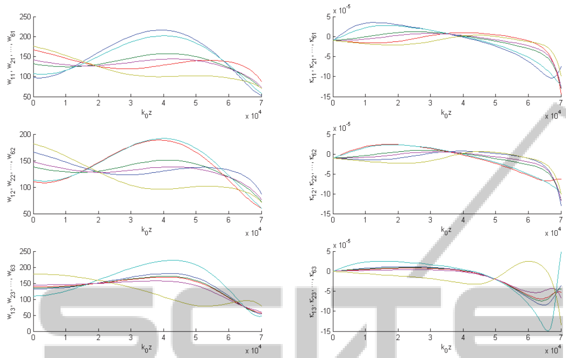

Figure 1: Evolution of 3D wave packets widths and chirps. Parameters: linear permittivities $\varepsilon_{0n}=2$, diagonal nonlinear permittivities $\varepsilon_{nn}=3\cdot10^{-5}$, off-diagonal nonlinear permittivities $\varepsilon_{nm}=9\cdot10^{-6}$, initial amplitudes: $|A_n(0)|=1$. Initial widths $w_{nl}(0)/\pi=$ [31, 53, 42; 42, 44, 43; 53, 35, 44; 34, 36, 35; 45, 47, 46; 56, 58, 57] are presented in Matlab notation (i.e. semicolons separate rows, colons separate elements in rows). Initial spatial chirps $\kappa_{1n}(0)=\kappa_{2n}(0)=-10^{-5}$ and temporal one $\kappa_{3n}(0)=0$, where $n\neq m$ for $n,m,l=1,2,\ldots,6$.

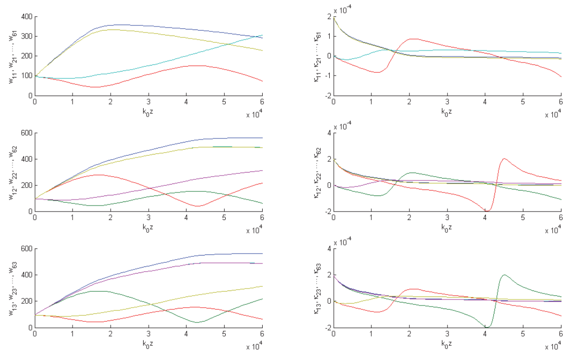

Figure 2: Evolution of 3D wave packets widths and chirps. Parameters: diagonal nonlinear permittivities $\varepsilon_{nn}=5.5\cdot10^{-4}$, initial spatial and temporal widths $w_{nk}(0)=30\pi$, initial spatial and temporal chirps $\kappa_{nl}(0)=$ [2, 2, 2; 0, 0, 2; 0, 2, 0; 0, 2, 2; 2, 0, 2; 2, 2, 0] $\cdot10^{-4}$, where $n\neq m$ for $n,m,l=1,2,\ldots,6$. The remaining parameters as in Figure 1.

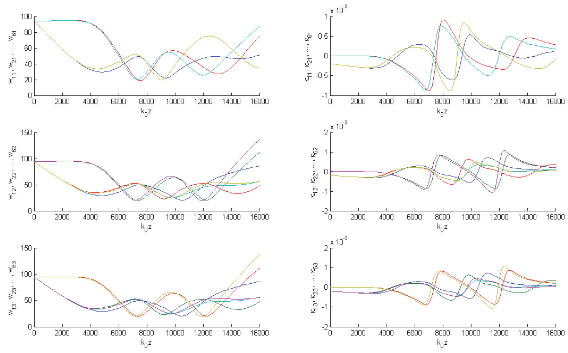

Figure 3: Evolution of 3D wave packets widths and chirps. Parameters: diagonal nonlinear permittivities $\varepsilon_{nn}=0.41339\cdot10^{-4}$, off-diagonal nonlinear permittivities $\varepsilon_{nm}=9\cdot10^{-6}$, initial spatial and temporal chirps $\kappa_{nl}(0)=$ [-2, -2, -2; 0, 0, -2; 0, -2, 0; 0, -2, -2; -2, 0, -2; -2, -2, 0] $\cdot10^{-4}$, where $n\neq m$ for $n,m,l=1,2,\ldots,6$. The remaining parameters as in Figure 1.

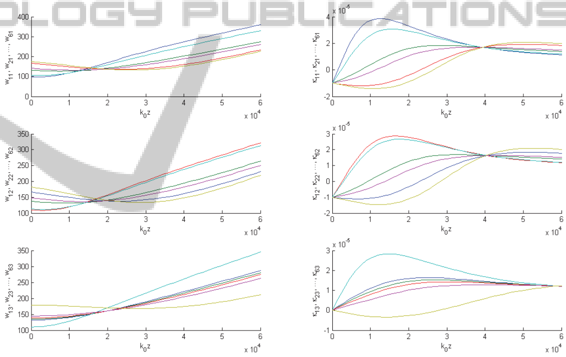

Figure 4: Evolution of 3D wave packets widths and chirps. Parameters: diagonal and off-diagonal nonlinear permittivities are equal $\varepsilon_{nn}=\varepsilon_{nm}=9\cdot10^{-6}$ for $n,m=1,2,\ldots,6$. The remaining parameters as in Figure 1.

to obtain a set of equations for the actual parameters

$$\frac{d}{dz'}w_{1j}(z')=w_{1j}\kappa_{1j}(z') \quad (41)$$

$$\frac{d}{dz'}\kappa_{1j}(z')=-\{\kappa_{1j}(z')\}^2+ \\ +\frac{1}{\sqrt{\varepsilon_{10}}}\left\{\left(\frac{1}{w_{1j}(z')}\right)^4-\frac{1}{2}\varepsilon_{ik}\left[\frac{|A_1(0)|}{w_{1j}(z')}\right]^2\frac{w_{11}(0)w_{12}(0)w_{13}(0)}{w_{11}w_{12}w_{13}}\right\} \quad (42)$$

identical with the ones derived using CGO method. However, we emphasize that authors of the paper (Jirauschek, Morgner, Kartner, 2002) made a lot of effort as compared with the simple and effective CGO procedure presented above. Moreover, generalization of variational procedure in Eq. (40) for N 3D wave packets in the form

$$\iiiint L\left(\begin{array}{l} u_1,u_1^*,\frac{\partial u_1}{\partial z},\frac{\partial u_1^*}{\partial z},\frac{\partial u_1}{\partial \eta_1},\frac{\partial u_1^*}{\partial \eta_1},\frac{\partial u_1}{\partial \eta_2},\frac{\partial u_1^*}{\partial \eta_2},\frac{\partial u_1}{\partial \tau},\frac{\partial u_1^*}{\partial \tau}, \\ u_2,u_2^*,\frac{\partial u_2}{\partial z},\frac{\partial u_2^*}{\partial z},\frac{\partial u_2}{\partial \eta_1},\frac{\partial u_2^*}{\partial \eta_1},\frac{\partial u_2}{\partial \eta_2},\frac{\partial u_2^*}{\partial \eta_2},\frac{\partial u_2}{\partial \tau},\frac{\partial u_2^*}{\partial \tau}, \\ \ldots, \\ u_N,u_N^*,\frac{\partial u_N}{\partial z},\frac{\partial u_N^*}{\partial z},\frac{\partial u_N}{\partial \eta_1},\frac{\partial u_N^*}{\partial \eta_1},\frac{\partial u_N}{\partial \eta_2},\frac{\partial u_N^*}{\partial \eta_2},\frac{\partial u_N}{\partial \tau},\frac{\partial u_N^*}{\partial \tau} \end{array}\right) dz d\eta_1 d\eta_2 d\tau \quad (43)$$

is very complicated. Numerical solutions of the set of Eqs. (38,39) are shown in Figs. (1-4) where we present the most interesting aspects of spatiotemporal evolution of a 3D wave packet propagating in a nonlinear medium. In these figures first of all we can notice some effects which cannot be obtained for single or pair of interacting wave packets. This way, in Fig. 1 we notice a specific multi-wave collapse effect. In Fig. 2, some of widths and chirps approach stationary case. The remaining ones still collapse. In Fig. 3 we notice oscillatory type of evolution, which cannot be achieved for one or two interacting wave packets. In the case of a number of interacting wave packets, we notice the tendency for the packets to imitate one another and approach one single state shown in Fig. 4 in the form of crossing plots of chirps.

5 CONCLUSIONS

The CGO was applied for spatiotemporal evolution of an arbitrary number of 3D mutually incoherent (with different carrier frequencies) Gaussian wave packets (GWPs) interacting and propagating in a nonlinear medium of Kerr type. The wavelength is short as compared to the overall size of the computational domain and direct numerical schemes, such as split-step fast Fourier method or finite differences beam propagation method, to solve a wave equation (Helmholtz and parabolic one) are very computationally expensive. The proposed approximation of geometrical optics with the complex generalization on complex eikonal and complex amplitude easily reduces the description of the propagation of beam, pulse and wave packet to solving complex ODEs. Numerical solving of ODE (the dependence on z) is much easier than solving partial differential equations (PDE, the dependence on x, y, z) for the same problem. CGO leads to the calculation of N times M points arranged along the z-axis. Other methods require the calculation of N times at $M \cdot K^2$ points, i.e. on a 3D mesh. With K equalling 100, the calculations can be up to 10,000-fold faster. This means that we obtain the results after 10 seconds in CGO instead of about 27 hours by other means. In this way, CGO method enables to perform very fast and efficient numerical simulations using commonly available computer numerical software like Matlab, Mathcad or Mathematica. CGO method is especially useful for engineers demanding a simpler method than those already used in nonlinear optics (variational approach (VA) and the method of moments (MM), which require the knowledge of Hamilton optics formalism). The numerical simulations performed in this paper show the efficiency of the CGO method on the example of a new sophisticated problem of nonlinear optics: interaction of an arbitrary number of 3D Gaussian wave packets propagating in a nonlinear (anomalous) dispersive medium of Kerr type. We demonstrate that the CGO method can describe also problems of fundamental optics more illustratively than the methods of Fourier transform and Fresnel diffraction integral. Complementary to the presented results, an on-line CGO solver is freely available at the authors' website: http://slawek.ps.pl/odelia.html. We can state that spatiotemporal CGO can be recognized to be the simplest and the most practical approach among commonly accepted methods of beam and fibre optics such as: spectral analysis, variational method, method of moments and method of generalized eikonal approximation.

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