An Analytical Approach to Evaluating Bivariate Functions of Fuzzy Numbers with One Local Extremum

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Abstract: This paper presents a novel analytical approach to evaluating continuous, bivariate functions of independent fuzzy numbers with one local extremum. The approach is based on a parametric α-cut representation of fuzzy numbers and allows for the inclusion of parameter uncertainties into mathematical models.

1 INTRODUCTION

There is an increasing effort in the scientific community to provide suitable methods for the inclusion of uncertainties into mathematical models. One way to do so is to introduce parametric uncertainty by representing the uncertain model parameters as fuzzy numbers (Dubois and Prade, 1980) and evaluating the model equations by means of Zadeh's extension principle (Zadeh, 1975). The evaluation of this *classical formulation* of the extension principle, however, turns out to be a highly complex task (Klimke, 2006). Fortunately, Buckley and Qu (1990) provide an *alternative formulation* that operates on α-cuts and is applicable to continuous functions of independent fuzzy numbers. Powerful *numerical* techniques have been developed to implement this alternative formulation (Moens and Hanss, 2011). However, there is no general *analytical* approach for a calculus with fuzzy numbers. For this purpose, a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers was introduced by the authors (Seibel and Schlattmann, 2013, 2014), which is based on the alternative formulation of the extension principle. In this paper, we extend this approach to bivariate functions of fuzzy numbers with one local extremum and no saddle points.

An outline of this paper is as follows. In Section 2, we give a definition of fuzzy numbers and present two important types. In Section 3, we briefly recall Zadeh's extension principle and introduce the alternative formulation based on α-cuts. In Section 4, we describe our analytical approach and give two illustrative examples. Finally, in Section 5, some conclusions are drawn.

2 FUZZY NUMBERS

Fuzzy numbers (Dubois and Prade, 1980) are a special class of fuzzy sets (Zadeh, 1965), which can be defined as follows.

A normal, convex fuzzy set $\tilde{x}$ over the real line $\mathbb{R}$ is called *fuzzy number* if there is exactly one $\bar{x} \in \mathbb{R}$ with $\mu_{\tilde{x}}(\bar{x}) = 1$ and the membership function is at least piecewise continuous. The value $\bar{x}$ is called the *modal* or *peak value* of $\tilde{x}$.

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications (Hanss, 2005). These typical fuzzy numbers shall be described in the following.

2.1 Triangular Fuzzy Numbers

Due to its very simple, linear membership function, the *triangular fuzzy number* (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} 1 + \dfrac{x - \bar{x}}{\tau^{\mathrm{L}}}, & \bar{x} - \tau^{\mathrm{L}} \leq x \leq \bar{x}, \\ 1 - \dfrac{x - \bar{x}}{\tau^{\mathrm{R}}}, & \bar{x} < x \leq \bar{x} + \tau^{\mathrm{R}}, \end{cases} \tag{1}$$

we use the parametric notation (Hanss, 2005)

$$\tilde{x} = \mathrm{tfn}(\bar{x}, \tau^{\mathrm{L}}, \tau^{\mathrm{R}}),$$

where $\bar{x}$ denotes the *modal value*, τ^{L} denotes the *left-hand*, and τ^{R} denotes the *right-hand spread* of $\tilde{x}$ (cf. Figure 1). If $\tau^{\mathrm{L}} = \tau^{\mathrm{R}}$, the TFN is called *symmetric*. Its α-cuts $x(\alpha) = \left[x^{\mathrm{L}}(\alpha), x^{\mathrm{R}}(\alpha)\right]$ result from the inverse

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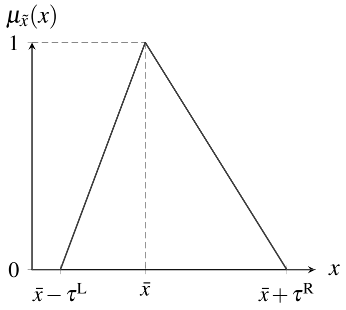


Figure 1: Triangular fuzzy number.

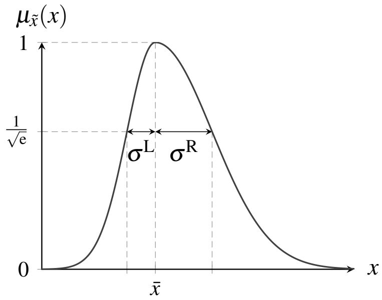


Figure 2: Gaussian fuzzy number.

functions of Eqs. (1) with respect to x:

$$x^{\mathrm{L}}(\alpha) = \bar{x} - \tau^{\mathrm{L}}(1-\alpha), \quad 0 < \alpha \leq 1,$$
$$x^{\mathrm{R}}(\alpha) = \bar{x} + \tau^{\mathrm{R}}(1-\alpha), \quad 0 < \alpha \leq 1.$$

2.2 Gaussian Fuzzy Numbers

Another widely-used fuzzy number in engineering is the *Gaussian fuzzy number* (GFN), which is based on the normal distribution from probability theory. In order to define a GFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} \exp\left[-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma^{\mathrm{L}}}\right)^2\right], & x \leq \bar{x}, \\ \exp\left[-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma^{\mathrm{R}}}\right)^2\right], & x > \bar{x}, \end{cases}$$

we use the parametric notation (Hanss, 2005)

$$\tilde{x} = \mathrm{gfn}(\bar{x}, \sigma^{\mathrm{L}}, \sigma^{\mathrm{R}}),$$

where $\bar{x}$ denotes the *modal value*, σ^{L} denotes the *left-hand*, and σ^{R} denotes the *right-hand standard deviation* of $\tilde{x}$ (cf. Figure 2). If $\sigma^{\mathrm{L}} = \sigma^{\mathrm{R}}$, the GFN is called *symmetric*. Its α-cuts $x(\alpha) = \left[x^{\mathrm{L}}(\alpha), x^{\mathrm{R}}(\alpha)\right]$ result in

$$x^{\mathrm{L}}(\alpha) = \bar{x} - \sigma^{\mathrm{L}}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \leq 1,$$
$$x^{\mathrm{R}}(\alpha) = \bar{x} + \sigma^{\mathrm{R}}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \leq 1.$$

3 EXTENSION PRINCIPLE

Zadeh's extension principle (Zadeh, 1975) allows for extending any real-valued function to a function of fuzzy numbers. More specifically, let $\tilde{x}_1, \ldots, \tilde{x}_n$ be n *independent* or *noninteractive* fuzzy numbers, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function with $y = f(x_1, \ldots, x_n)$. The fuzzy extension $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$ is then defined by

$$\mu_{\tilde{y}}(y) = \sup_{y=f(x_1,\ldots,x_n)} \min\{\mu_{\tilde{x}_1}(x_1), \ldots, \mu_{\tilde{x}_n}(x_n)\}.$$

In case of *interdependency* between $\tilde{x}_1, \ldots, \tilde{x}_n$, the minimum operator should be replaced by a suitable *triangular norm* (Scheerlinck, 2011). In this paper, however, we restrict ourselves to independent fuzzy numbers.

The evaluation of this *classical formulation* of the extension principle turns out to be a highly complex task (Klimke, 2006). Fortunately, Buckley and Qu (1990) provide an *alternative formulation* that operates on α-cuts:

Let $x_1(\alpha), \ldots, x_n(\alpha)$ denote the α-cuts of the n independent fuzzy numbers $\tilde{x}_1, \ldots, \tilde{x}_n$, and let f be continuous. Then, the α-cuts $y(\alpha) = \left[y^{\mathrm{L}}(\alpha), y^{\mathrm{R}}(\alpha)\right]$ of $\tilde{y}$ can be computed from

$$y^{\mathrm{L}}(\alpha) = \min\{f(x_1, \ldots, x_n) \,|\, (x_1, \ldots, x_n) \in \Omega(\alpha)\},$$
$$y^{\mathrm{R}}(\alpha) = \max\{f(x_1, \ldots, x_n) \,|\, (x_1, \ldots, x_n) \in \Omega(\alpha)\},$$

where $\Omega(\alpha) = x_1(\alpha) \times \cdots \times x_n(\alpha)$ represent the n-dimensional interval boxes that are spanned by the α-cuts $x_1(\alpha), \ldots, x_n(\alpha)$.

If the continuous function f is (strictly) monotonic increasing in x_i, $i = 1, \ldots, k$, and (strictly) monotonic decreasing in x_j, $j = 1, \ldots, \ell$, in the domain of interest, and if $k + \ell = n$, then, the minimum values of f inside of every sub-domain $\Omega(\alpha)$ are always found at the left boundaries of $x_i(\alpha)$ and the right boundaries of $x_j(\alpha)$, and its maximum values at the right boundaries of $x_i(\alpha)$ and the left boundaries of $x_j(\alpha)$, respectively. In such case, the α-cuts $y(\alpha) = \left[y^{\mathrm{L}}(\alpha), y^{\mathrm{R}}(\alpha)\right]$ of $\tilde{y}$ become (Seibel and Schlattmann, 2013)

$$\begin{aligned} y^{\mathrm{L}}(\alpha) &= f\left(x_i^{\mathrm{L}}(\alpha), x_j^{\mathrm{R}}(\alpha)\right), \quad 0 < \alpha \leq 1, \\ y^{\mathrm{R}}(\alpha) &= f\left(x_i^{\mathrm{R}}(\alpha), x_j^{\mathrm{L}}(\alpha)\right), \quad 0 < \alpha \leq 1, \end{aligned} \tag{2}$$

with $x_m(\alpha) = \left[x_m^{\mathrm{L}}(\alpha), x_m^{\mathrm{R}}(\alpha)\right]$, $m = 1, \ldots, n$. If Eqs. (2) are invertible with respect to α, then the membership function of $\tilde{y}$ yields (Seibel and Schlattmann, 2013)

$$\mu_{\tilde{y}}(y) = \begin{cases} y^{\mathrm{L}}(\alpha)^{-1}, & y^{\mathrm{L}}(0) < y \leq y^{\mathrm{L}}(1), \\ y^{\mathrm{R}}(\alpha)^{-1}, & y^{\mathrm{R}}(1) < y < y^{\mathrm{R}}(0). \end{cases}$$

The analytical approach, which is presented in the next section, is based on this alternative formulation of the extension principle.

4 ANALYTICAL APPROACH

For evaluating continuous, nonmonotonic functions of independent fuzzy numbers, Dong and Shah (1987) and Fortin et al. (2008) suggest to include the extreme points as constant profiles into the computation. However, this is not enough and can lead to erroneous results, as was pointed out by Wood et al. (1992). More specifically, all permutations of the interval boundaries of $x_m(\alpha)$, $m = 1,\ldots,n$, with the components of the extreme points have to be considered as well.

Let f be a continuous, bivariate function with positive arguments and no saddle points. If $\boldsymbol{x}^* = (x_1^*, x_2^*)$ is the only maximum (minimum) of f, then $f(\boldsymbol{x}-\boldsymbol{x}^*)$ is (strictly) monotonic decreasing (increasing) in x_1 and x_2. In order to compute the α-cuts $y(\alpha) = \left[y^{\mathrm{L}}(\alpha), y^{\mathrm{R}}(\alpha)\right]$ of $\tilde{y}$, we distinguish between the following two scenarios.

4.1 $\boldsymbol{x}^*$ is a Maximum

The modal point $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2)$ divides the (positive) domain of interest into four subdomains. The analytical solution depends in which subdomain the maximum is located. Let $\alpha_1^* = \mu_{\tilde{x}_1}(x_1^*)$ and $\alpha_2^* = \mu_{\tilde{x}_2}(x_2^*)$, then

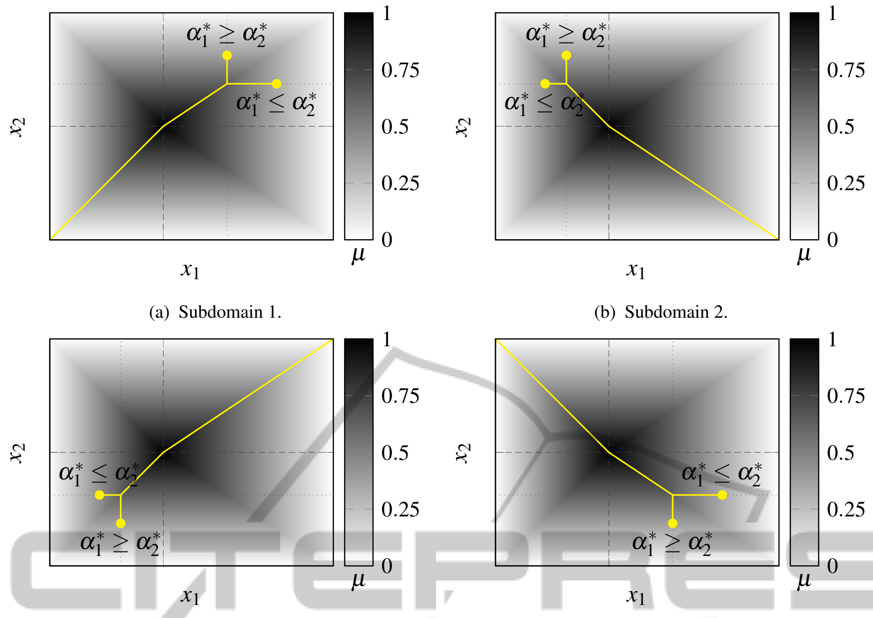
1. $x_1^* \geq \bar{x}_1$ and $x_2^* \geq \bar{x}_2$:
$$y^{\mathrm{L}}(\alpha) = f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), \quad 0 < \alpha \leq 1,$$
   (a) $\alpha_1^* \geq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^*, x_2^{\mathrm{R}}(\alpha)\big), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), & \alpha_1^* < \alpha \leq 1, \end{cases}$$
   (b) $\alpha_1^* \leq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^*\big), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), & \alpha_2^* < \alpha \leq 1, \end{cases}$$
2. $x_1^* \leq \bar{x}_1$ and $x_2^* \geq \bar{x}_2$:
$$y^{\mathrm{L}}(\alpha) = f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), \quad 0 < \alpha \leq 1,$$
   (a) $\alpha_1^* \geq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^*, x_2^{\mathrm{R}}(\alpha)\big), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), & \alpha_1^* < \alpha \leq 1, \end{cases}$$
   (b) $\alpha_1^* \leq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^*\big), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), & \alpha_2^* < \alpha \leq 1, \end{cases}$$
3. $x_1^* \leq \bar{x}_1$ and $x_2^* \leq \bar{x}_2$:
$$y^{\mathrm{L}}(\alpha) = f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), \quad 0 < \alpha \leq 1,$$
   (a) $\alpha_1^* \geq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^*, x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_1^* < \alpha \leq 1, \end{cases}$$
   (b) $\alpha_1^* \leq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^*\big), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq 1, \end{cases}$$
4. $x_1^* \geq \bar{x}_1$ and $x_2^* \leq \bar{x}_2$:
$$y^{\mathrm{L}}(\alpha) = f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), \quad 0 < \alpha \leq 1,$$
   (a) $\alpha_1^* \geq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^*, x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_1^* < \alpha \leq 1, \end{cases}$$
   (b) $\alpha_1^* \leq \alpha_2^*$:
$$y^{\mathrm{R}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^*\big), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq 1. \end{cases}$$

Note that $f(x_1^*, x_2^*)$ is only necessary if (x_1^*, x_2^*) lies within the domain of interest.

The solution paths of $y(\alpha)$ for $\tilde{x}_1$ and $\tilde{x}_2$ being of triangular type are illustrated in Figure 3. We can see that, starting from the extremum, which is marked by the yellow circle, the modal point acts as an attractor and pulls the solution parallel to a coordinate axis directly into one of the two principal diagonals running from the lower-left to the upper-right corner and from the lower-right to the upper-left corner of the domain $\mathrm{supp}(\tilde{x}_1) \times \mathrm{supp}(\tilde{x}_2)$, respectively.

Example 1. The function $f_1 : \mathbb{R}_+^2 \to \mathbb{R}$ with
$$y_1 = f_1(x_1, x_2) = -x_1^2 - x_2^2 + 5x_1 + x_2$$
shall be evaluated for the two fuzzy numbers $\tilde{x}_1 = \mathrm{tfn}(2,2,3)$ and $\tilde{x}_2 = \mathrm{tfn}(2,2,2)$.

Since f_1 has only one maximum at $\boldsymbol{x}^* = (2.5, 0.5)$, $\boldsymbol{x}^*$ is located in subdomain 4, and $\alpha_1^* = 0.8\bar{3} \geq 0.25 = \alpha_2^*$, the α-cuts $y_1(\alpha) = \left[y_1^{\mathrm{L}}(\alpha), y_1^{\mathrm{R}}(\alpha)\right]$ of $\tilde{y}_1$ are
$$y_1^{\mathrm{L}}(\alpha) = -8\alpha^2 + 24\alpha - 12, \quad 0 < \alpha \leq 1,$$
$$y_1^{\mathrm{R}}(\alpha) = \begin{cases} 6.5, & 0 < \alpha \leq 0.25, \\ -4\alpha^2 + 2\alpha + 6.25, & 0.25 < \alpha \leq 0.8\bar{3}, \\ -13\alpha^2 + 17\alpha, & 0.8\bar{3} < \alpha \leq 1. \end{cases}$$



(a) Subdomain 1.
(b) Subdomain 2.
(c) Subdomain 3.
(d) Subdomain 4.

Figure 3: Solution paths of $y(\alpha)$.

With $y_1^{\mathrm{L}}(0) = -12$, $y_1^{\mathrm{L}}(1) = 4 = y_1^{\mathrm{R}}(1)$, $y_1^{\mathrm{R}}(0.8\bar{3}) \approx 5.1$, and $y_1^{\mathrm{R}}(0.25) = 6.5$, the membership function of $\tilde{y}_1$ yields

$$\mu_{\tilde{y}_1}(y) = \begin{cases} \frac{3}{2} - \frac{1}{4}\sqrt{12-2y}, & -12 < y \leq 4, \\ \frac{17}{26} + \frac{1}{26}\sqrt{289-52y}, & 4 < y \leq 5.1, \\ \frac{1}{4} + \frac{1}{4}\sqrt{26-4y}, & 5.1 < y < 6.5. \end{cases}$$

The plot of $\mu_{\tilde{y}_1}(y)$ is shown in Figure 4.

4.2 x^* is a Minimum

The modal point $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2)$ divides the (positive) domain of interest into four subdomains. The analytical solution depends in which subdomain the minimum is located. Let $\alpha_1^* = \mu_{\tilde{x}_1}(x_1^*)$ and $\alpha_2^* = \mu_{\tilde{x}_2}(x_2^*)$, then

1. $x_1^* \geq \bar{x}_1$ and $x_2^* \geq \bar{x}_2$:

$$y^{\mathrm{R}}(\alpha) = f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \quad 0 < \alpha \leq 1,$$

(a) $\alpha_1^* \geq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_2^*, \\ f\left(x_1^*, x_2^{\mathrm{R}}(\alpha)\right), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), & \alpha_1^* < \alpha \leq 1, \end{cases}$$

(b) $\alpha_1^* \leq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{R}}(\alpha), x_2^*\right), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), & \alpha_2^* < \alpha \leq 1, \end{cases}$$

2. $x_1^* \leq \bar{x}_1$ and $x_2^* \geq \bar{x}_2$:

$$y^{\mathrm{R}}(\alpha) = f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \quad 0 < \alpha \leq 1,$$

(a) $\alpha_1^* \geq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_2^*, \\ f\left(x_1^*, x_2^{\mathrm{R}}(\alpha)\right), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), & \alpha_1^* < \alpha \leq 1, \end{cases}$$

(b) $\alpha_1^* \leq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^*\right), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), & \alpha_2^* < \alpha \leq 1, \end{cases}$$

3. $x_1^* \leq \bar{x}_1$ and $x_2^* \leq \bar{x}_2$:

$$y^{\mathrm{R}}(\alpha) = f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), \quad 0 < \alpha \leq 1,$$

(a) $\alpha_1^* \geq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_2^*, \\ f\left(x_1^*, x_2^{\mathrm{L}}(\alpha)\right), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), & \alpha_1^* < \alpha \leq 1, \end{cases}$$

(b) $\alpha_1^* \leq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\left(x_1^*, x_2^*\right), & 0 < \alpha \leq \alpha_1^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^*\right), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), & \alpha_2^* < \alpha \leq 1, \end{cases}$$

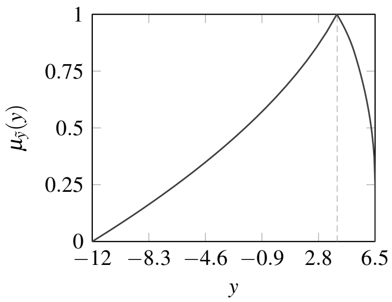


Figure 4: Membership function of $\tilde{y}_1$.

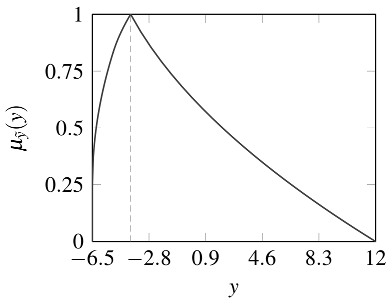


Figure 5: Membership function of $\tilde{y}_2$.

4. $x_1^* \geq \bar{x}_1$ and $x_2^* \leq \bar{x}_2$:

$$y^{\mathrm{R}}(\alpha) = f\big(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\big), \quad 0 < \alpha \leq 1,$$

(a) $\alpha_1^* \geq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_2^*, \\ f\big(x_1^*, x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_1^* < \alpha \leq 1, \end{cases}$$

(b) $\alpha_1^* \leq \alpha_2^*$:

$$y^{\mathrm{L}}(\alpha) = \begin{cases} f\big(x_1^*, x_2^*\big), & 0 < \alpha \leq \alpha_1^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^*\big), & \alpha_1^* < \alpha \leq \alpha_2^*, \\ f\big(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\big), & \alpha_2^* < \alpha \leq 1. \end{cases}$$

Note that, again, $f(x_1^*, x_2^*)$ is only necessary if (x_1^*, x_2^*) lies within the domain of interest.

In this scenario, the solution paths of $y(\alpha)$ are the same as those in Figure 3.

Example 2. Now, the function $f_2\colon \mathbb{R}_+^2 \to \mathbb{R}$ with

$$y_2 = f_2(x_1, x_2) = -f_1(x_1, x_2) = x_1^2 + x_2^2 - 5x_1 - x_2$$

shall be evaluated for the two fuzzy numbers from Example 1.

Since f_2 has only one minimum at $\boldsymbol{x}^* = (2.5, 0.5)$, $\boldsymbol{x}^*$ is located in subdomain 4, and $\alpha_1^* = 0.8\bar{3} \geq 0.25 = \alpha_2^*$, the α-cuts $y_2(\alpha) = \big[y_2^{\mathrm{L}}(\alpha), y_2^{\mathrm{R}}(\alpha)\big]$ of $\tilde{y}_2$ are

$$y_2^{\mathrm{L}}(\alpha) = \begin{cases} -6.5, & 0 < \alpha \leq 0.25, \\ 4\alpha^2 - 2\alpha - 6.25, & 0.25 < \alpha \leq 0.8\bar{3}, \\ 13\alpha^2 - 17\alpha, & 0.8\bar{3} < \alpha \leq 1, \end{cases}$$

$$y_2^{\mathrm{R}}(\alpha) = 8\alpha^2 - 24\alpha + 12, \quad 0 < \alpha \leq 1.$$

With $y_2^{\mathrm{L}}(0.25) = -6.5$, $y_2^{\mathrm{L}}(0.8\bar{3}) \approx -5.1$, $y_2^{\mathrm{L}}(1) = -4 = y_2^{\mathrm{R}}(1)$, and $y_2^{\mathrm{R}}(0) = 12$, the membership function of $\tilde{y}_2$ yields

$$\mu_{\tilde{y}_2}(y) = \begin{cases} \frac{1}{4} + \frac{1}{4}\sqrt{4y + 26}, & -6.5 < y \leq -5.1, \\ \frac{17}{26} + \frac{1}{26}\sqrt{52y + 289}, & -5.1 < y \leq -4, \\ \frac{3}{2} - \frac{1}{4}\sqrt{2y + 12}, & -4 < y < 12. \end{cases}$$

The plot of $\mu_{\tilde{y}_2}(y)$ is shown in Figure 5. Note that $\mu_{\tilde{y}_7}(y)$ and $\mu_{\tilde{y}_8}(y)$ are symmetric to each other.

5 CONCLUSIONS

We extended our analytical approach from Seibel and Schlattmann (2013, 2014) to bivariate functions of fuzzy numbers with one local extremum and no saddle points. It is based on an alternative formulation of the extension principle and allows for the inclusion of parameter uncertainties into mathematical models. Using the patterns from Figure 3, this approach can be easily extended to bivariate functions with more than one extremum, see Wood et al. (1992) and Degrauwe (2007) for similar numerical approaches.

An analytical solution has the advantage that the degrees of membership of the fuzzy output can be computed for any value within the support, whereas a numerical solution only provides a finite number of values. Furthermore, our approach also allows a symbolic processing of uncertainties.

In further research activities, this approach shall be extended to general, nonmonotonic functions of independent fuzzy numbers, where the influence of interdependency shall be investigated as well.

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