Interactive Fuzzy Decision Making for Multiobjective Fuzzy Random Linear Programming Problems

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Abstract: In this paper, we propose an interactive fuzzy decision making method for multiobjective fuzzy random linear programming problems (MOFRLP), in which the criteria of probability maximization and fractile optimization are considered simultaneously. In the proposed method, it is assumed that the decision maker has fuzzy goals for not only objective functions of MOFRLP but also permissible probability levels in a fractile optimization model for MOFRLP, and such fuzzy goals are quantified by eliciting the corresponding membership functions. Using the fuzzy decision, such two kinds of membership functions are integrated. In the integrated membership space, the satisfactory solution is obtained from among a Pareto optimal solution set through the interaction with the decision maker.

1 INTRODUCTION

In the real world decision making situations, we often have to make a decision under uncertainty. In order to deal with decision problems involving uncertainty, stochastic programming approaches (Birge and Louveaux, 1997; Charnes and Cooper, 1959; Dantzig, 1955; Kall and Mayer, 2005) and fuzzy programming approaches (Lai and Hwang, 1992; Sakawa, 1993; Zimmermann, 2011) have been developed. Recently, mathematical programming problems with fuzzy random variables (Kwakernaak, 1978) have been proposed (Katagiri et al., 1997; Luhandjula and Gupta, 1996; Wang and Qiao, 1993) whose concept includes both probabilistic uncertainty and fuzzy ones simultaneously. For multiobjective fuzzy random linear programming problems (MOFRLP), (Sakawa et al., 2011) formulated and proposed interactive methods to obtain the satisfactory solution. In their methods, it is required in advance for the decision maker to specify permissible possibility levels in a probability maximization model or permissible probability levels in a fractile optimization model. However, it seems to be very difficult for the decision maker to specify such permissible levels appropriately. From such a point of view, (Yano and Matsui, 2011) have proposed a fuzzy approach for MOFRLP, in which the decision maker specifies the membership functions for the fuzzy goals of both objective functions of MOFRLP and permissible probability levels. In the proposed method, it is assumed that the decision maker adopts the fuzzy decision (Sakawa, 1993) to integrate the membership functions. However, the fuzzy decision can be viewed as one special operator to integrate the membership functions. If the decision maker would not adopt the fuzzy decision, the proposed method cannot be applied in the real-world decision situation. In this paper, we propose an interactive fuzzy decision making method for MOFRLP to obtain the satisfactory solution from among a Pareto optimal solution set. In section 2, MOFRLP is formulated by using a concept of a possibility measure (Dubois and Prade, 1980). In section 3, through a probability maximization model, the D_p-Pareto optimal concept is introduced in order to deal with MOFRLP, and the minmax problem is formulated to obtain a D_p-Pareto optimal solution, which can be solved on the basis of the linear programming technique. In section 4, through a fractile optimization model, the D_G-Pareto optimal concept is introduced and the minmax problem is formulated to obtain a D_G-Pareto optimal solution. In section 5, we propose an interactive algorithm to obtain the satisfactory solution from among a Pareto optimal solution set by solving the minmax problem on the ba-

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sis of the linear programming technique. In section 5, in order to demonstrate the interactive processes under the hypothetical decision maker, a two-objective fuzzy random linear programming problem, as a numerical example, is formulated and solved by using the proposed interactive algorithm. Finally, in section 7, we conclude this paper.

2 MULTIOBJECTIVE FUZZY RANDOM LINEAR PROGRAMMING PROBLEMS

In this section, we focus on multiobjective programming problems involving fuzzy random variable coefficients in objective functions, which is called multiobjective fuzzy random linear programming problem (MOFRLP).

[MOFRLP]

$$\min \tilde{\bar{C}}x = (\tilde{\bar{c}}_1 x, \cdots, \tilde{\bar{c}}_k x)$$

subject to

$$x \in X \overset{\text{def}}{=} \{x \in \mathrm{R}^n \mid Ax \le b, x \ge 0\}$$

where $x = (x_1, x_2, \cdots, x_n)^T$ is an n dimensional decision variable column vector, A is an $(m \times n)$ coefficient matrix, $b = (b_1, \cdots, b_m)^T$ is an m dimensional column vector. $\tilde{\bar{c}}_i = (\tilde{\bar{c}}_{i1}, \cdots, \tilde{\bar{c}}_{in}), i = 1, \cdots, k$, are coefficient vectors of objective function $\tilde{\bar{c}}_i x$, whose elements are fuzzy random variables (Kwakernaak, 1978; Puri and Ralescu, 1986; Sakawa et al., 2011), and the symbols "-" and "~" mean randomness and fuzziness respectively.

In order to deal with the objective functions $\tilde{\bar{c}}_i x, i = 1, \cdots, k$, (Sakawa et al., 2011) proposed an LR-type fuzzy random variable which can be regarded as a special version of a fuzzy random variable. Under the occurrence of each elementary event ω, $\tilde{\bar{c}}_{ij}(\omega)$ is a realization of an LR-type fuzzy random variable $\tilde{\bar{c}}_{ij}$, which is an LR fuzzy number (Dubois and Prade, 1980) whose membership function is defined as follows.

$$\mu_{\tilde{\bar{c}}_{ij}(\omega)}(s) = \begin{cases} L\left(\frac{\bar{d}_{ij}(\omega)-s}{\bar{\alpha}_{ij}(\omega)}\right) & (s \le \bar{d}_{ij}(\omega)\ \forall\omega), \\ R\left(\frac{s-\bar{d}_{ij}(\omega)}{\bar{\beta}_{ij}(\omega)}\right) & (s > \bar{d}_{ij}(\omega)\ \forall\omega), \end{cases}$$

where the function $L(t) \overset{\text{def}}{=} \max\{0, l(t)\}$ is a real-valued continuous function from $[0, \infty)$ to $[0, 1]$, and $l(t)$ is a strictly decreasing continuous function satisfying $l(0) = 1$. Also, $R(t) \overset{\text{def}}{=} \max\{0, r(t)\}$ satisfies the same conditions. $\bar{d}_{ij}, \bar{\alpha}_{ij}, \bar{\beta}_{ij}$ are random variables expressed by $\bar{d}_{ij} = d^1_{ij} + \bar{t}_i d^2_{ij}$, $\bar{\alpha}_{ij} = \alpha^1_{ij} + \bar{t}_i \alpha^2_{ij}$ and $\bar{\beta}_{ij} = \beta^1_{ij} + \bar{t}_i \beta^2_{ij}$. $\bar{t}_i$ is a random variable whose distribution function is denoted by $T_i(\cdot)$ which is strictly increasing and continuous, and $d^1_{ij}, d^2_{ij}, \alpha^1_{ij}, \alpha^2_{ij}, \beta^1_{ij}, \beta^2_{ij}$ are constants.

(Sakawa et al., 2011) transformed MOFRLP into a multiobjective stochastic programming problem (MOSP) by using a concept of a possibility measure (Dubois and Prade, 1980). As shown in (Sakawa et al., 2011), the realizations $\tilde{\bar{c}}_i(\omega)x$ becomes an LR fuzzy number characterized by the following membership functions on the basis of the extension principle (Dubois and Prade, 1980).

$$\mu_{\tilde{\bar{c}}_i(\omega)x}(y) = \begin{cases} L\left(\frac{\bar{d}_i(\omega)x-y}{\bar{\alpha}_i(\omega)x}\right) & y \le \bar{d}_i(\omega)x \\ R\left(\frac{y-\bar{d}_i(\omega)x}{\bar{\beta}_i(\omega)x}\right) & y > \bar{d}_i(\omega)x \end{cases}$$

For the realizations $\tilde{\bar{c}}_i(\omega)x, i = 1, \cdots, k$, it is assumed that the decision maker has fuzzy goals $\tilde{G}_i, i = 1, \cdots, k$ (Sakawa, 1993), whose membership functions $\mu_{\tilde{G}_i}(y)$, $i = 1, \cdots, k$ are continuous and strictly decreasing for minimization problems. By using a concept of a possibility measure (Dubois and Prade, 1980), a degree of possibility that the objective function value $\tilde{\bar{c}}_i x$ satisfies the fuzzy goal $\tilde{G}_i$ is expressed as follows (Katagiri et al., 1997).

$$\Pi_{\tilde{\bar{c}}_i x}(\tilde{G}_i) \overset{\text{def}}{=} \sup_y \min\{\mu_{\tilde{\bar{c}}_i x}(y), \mu_{\tilde{G}_i}(y)\} \qquad (1)$$

Using a possibility measure, MOFRLP can be transformed into the following multiobjective stochastic programming problem (MOSP).

[MOSP]

$$\max_{x \in X} (\Pi_{\tilde{\bar{c}}_1 x}(\tilde{G}_1), \cdots, \Pi_{\tilde{\bar{c}}_k x}(\tilde{G}_k)) \qquad (2)$$

(Sakawa et al., 2011) transformed MOSP into the usual multiobjective programming problems through a probability maximization model and a fractile maximization model, and proposed interactive algorithms to obtain a satisfactory solution. In their methods, the decision maker must specify permissible probability levels or permissible possibility levels for the objective functions in advance. However, it seems to be very difficult to specify appropriate permissible levels because they have a great influence on the objective function values or distribution function values. In the following sections, by assuming that the decision maker has fuzzy goals for permissible probability levels and permissible possibility levels, we propose an interactive fuzzy decision making method for MOFRLP to obtain a satisfactory solution.

3 A FORMULATION THROUGH A PROBABILITY MAXIMIZATION MODEL

For the objective function of MOSP, if the decision maker specifies the permissible possibility level $h_i \in [0,1]$, then MOSP can be formulated as the following multiobjective programming problem through a probability maximization model.

[MOP1(h)]

$$\max_{x\in X} \quad (\Pr(\omega \mid \Pi_{\tilde{\bar{C}}_1(\omega)x}(\tilde{G}_1) \geq h_1), \cdots, \\ \Pr(\omega \mid \Pi_{\tilde{\bar{C}}_k(\omega)x}(\tilde{G}_k) \geq h_k))$$

where $\Pr(\cdot)$ is a probability measure, $h = (h_1, \cdots, h_k)$ is a vector of permissible possibility levels. In MOP1(h), the inequality $\Pi_{\tilde{\bar{C}}_i(\omega)x}(\widetilde{G}_i) \geq h_i$ can be equivalently transformed into the following form.

$$\sup_y \min\{\mu_{\tilde{\bar{C}}_i x}(y), \mu_{\tilde{G}_i}(y)\} \geq h_i,$$
$$\Leftrightarrow \quad (\bar{d}_i(\omega) - L^{-1}(h_i)\bar{\alpha}_i(\omega))x \leq \mu^{-1}_{\tilde{G}_i}(h_i)$$

where $L^{-1}(\cdot)$ and $R^{-1}(\cdot)$ are pseudo-inverse functions. Therefore, using the distribution function $T_i(\cdot)$ of the random variable $\bar{t}_i$, the objective functions in MOP1(h) can be expressed as the following form.

$$\begin{aligned} & \Pr(\omega \mid \Pi_{\tilde{\bar{C}}_i(\omega)x}(\tilde{G}_i) \geq h_i) \\ = \quad & T_i\left(\frac{\mu^{-1}_{\tilde{G}_i}(h_i) - (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x)}{d_i^2 x - L^{-1}(h_i)\alpha_i^2 x}\right) \\ \stackrel{\text{def}}{=} \quad & p_i(x, h_i) \end{aligned} \tag{3}$$

where it is assumed that $(d_i^2 - L^{-1}(0)\alpha_i^2)x > 0$, $i = 1, \cdots, k$ for any $x \in X$. As a result, using $p_i(x, h_i), i = 1, \cdots, k$, MOP1$(h)$ can be transformed to the following simple form (Sakawa et al., 2011).

[MOP2(h)]

$$\max_{x\in X}(p_1(x, h_1), \cdots, p_k(x, h_k))$$

In MOP2(h), the decision maker seems to prefer not only the larger value of a permissible possibility level h_i but also the larger value of the corresponding distribution function $p_i(x, h_i)$. Since these values conflict with each other, the larger value of a permissible possibility level h_i results in the less value of the corresponding distribution function $p_i(x, h_i)$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP2(h).

[MOP3]

$$\max_{x\in X, h_i\in[0,1], i=1,\cdots,k} \quad (p_1(x, h_1), \cdots, p_k(x, h_k), \\ h_1, \cdots, h_k)$$

It should be noted in MOP3 that permissible possibility levels $h_i, i = 1, \cdots, k$ are not the fixed values but the decision variables. Considering the imprecise nature of the decision maker's judgment, it is natural to assume that the decision maker has fuzzy goals for $p_i(x, h_i), i = 1, \cdots, k$. In this section, we assume that such fuzzy goals can be quantified by eliciting the corresponding membership functions. Let us denote a membership function of a distribution function as $\mu_{p_i}(p_i(x, h_i))$. Then, MOP3 can be transformed to the following multiobjective programming problem.

[MOP4]

$$\max_{x\in X, h_i\in[0,1], i=1,\cdots,k} \quad (\mu_{p_1}(p_1(x, h_1)), \cdots, \\ \mu_{p_k}(p_k(x, h_k)), h_1, \cdots, h_k)$$

In order to elicit the membership functions $\mu_{p_i}(p_i(x, h_i)), i = 1, \cdots, k$ appropriately, we suggest the following procedures. First of all, the decision maker sets the intervals $H_i = [h_{i\min}, h_{i\max}]$ for permissible possibility levels, where $h_{i\min}$ is a maximum value of an unacceptable levels and $h_{i\max}$ is a minimum value of a sufficiently satisfactory levels. For the interval H_i, the corresponding interval of $p_i(x, \hat{h}_i)$ can be defined as $P_i(H_i) = [p_{i\min}, p_{i\max}] = \{p_i(x, h_i) \mid x \in X, h_i \in H_i\}$. $p_{i\max}$ can be obtained by solving the following optimization problem.

$$p_{i\max} \stackrel{\text{def}}{=} \max_{x\in X} p_i(x, h_{i\min}) \tag{4}$$

In order to obtain $p_{i\min}$, we first solve the optimization problems $\max_{x\in X} p_i(x, h_{i\max}), i = 1, \cdots, k$, and denote the corresponding optimal solutions as $x_i, i = 1, \cdots, k$. Using the optimal solution $x_i, i = 1, \cdots, k$, $p_{i\min}$ can be obtained as the following minimum value.

$$p_{i\min} \stackrel{\text{def}}{=} \min_{\ell=1,\cdots,k, \ell\neq i} p_i(x_\ell, h_{i\max}) \tag{5}$$

For the membership functions $\mu_{p_i}(p_i(x, h_i)), i = 1, \cdots, k$ defined on $P_i(H_i)$, we make the following assumption.

Assumption 1.
$\mu_{p_i}(p_i(x, h_i)), i = 1, \cdots, k$ are strictly increasing and continuous with respect to $p_i(x, h_i) \in P_i(H_i)$, and $\mu_{p_i}(p_{i\min}) = 0$, $\mu_{p_i}(p_{i\max}) = 1$.

It should be noted here that $\mu_{p_i}(p_i(x, h_i))$ is strictly decreasing with respect to $h_i \in H_i$. If the decision maker adopts the fuzzy decision (Sakawa, 1993) to integrate $\mu_{p_i}(p_i(x, h_i))$ and h_i, MOP4 can be transformed into the following form.

[MOP5]

$$\max_{x\in X, h_i=H_i, i=1,\cdots,k} \left(\mu_{D_{p_1}}(x, h_1), \cdots, \mu_{D_{p_k}}(x, h_k)\right)$$

where

$$\mu_{D_{p_i}}(x, h_i) \stackrel{\text{def}}{=} \min\{h_i, \mu_{p_i}(p_i(x, h_i))\} \tag{6}$$

In order to deal with MOP5, we introduce a D_p-Pareto optimal solution concept.

Definition 1.

$x^* \in X, h_i^* \in H_i, i = 1, \cdots, k$ is said to be a D_p-Pareto optimal solution to MOP5, if and only if there does not exist another $x \in X, h_i \in H_i, i = 1, \cdots, k$ such that $\mu_{D_{p_i}}(x, h_i) \geq \mu_{D_{p_i}}(x^*, h_i^*)$ $i = 1, \cdots, k$ with strict inequality holding for at least one i.

For generating a candidate of a satisfactory solution which is also D_p-Pareto optimal, the decision maker is asked to specify the reference membership values (Sakawa, 1993) in membership space. Once the reference membership values $\hat{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ are specified, the corresponding D_p-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX1($\hat{\mu}$)]

$$\min_{x \in X, h_i \in H_i, i=1,\cdots,k, \lambda \in \Lambda} \lambda \tag{7}$$

subject to

$$\hat{\mu}_i - \mu_{p_i}(p_i(x, h_i)) \leq \lambda, i = 1, \cdots, k \tag{8}$$

$$\hat{\mu}_i - h_i \leq \lambda, i = 1, \cdots, k \tag{9}$$

where

$$\Lambda = [\max_{i=1,\cdots,k} \hat{\mu}_i - 1, \min_{i=1,\cdots,k} \hat{\mu}_i]. \tag{10}$$

From Assumption 1, the inequality constraints (8) can be transformed into the following form.

$$\begin{aligned} & \hat{\mu}_i - \mu_{p_i}(p_i(x, h_i)) \leq \lambda \\ \Leftrightarrow \quad & \mu_{\tilde{G}_i}^{-1}(h_i) \geq (d_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) d_i^2 x) \\ & -L^{-1}(h_i)(\alpha_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \alpha_i^2 x) \end{aligned} \tag{11}$$

In (11), because of $\hat{\mu}_i - \lambda \leq h_i$ and Assumption 1, it holds that $\mu_{\tilde{G}_i}^{-1}(h_i) \leq \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda)$ and $L^{-1}(h_i) \leq L^{-1}(\hat{\mu}_i - \lambda)$. Since it is guaranteed that $(\alpha_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda))\ \alpha_i^2 x) > 0$, the following inequalities can be derived.

$$\begin{aligned} & (d_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) d_i^2 x) \\ & -L^{-1}(h_i)(\alpha_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \alpha_i^2 x) \\ \geq \quad & (d_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) d_i^2 x) \\ & -L^{-1}(\hat{\mu}_i - \lambda)(\alpha_i^1 x + T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \alpha_i^2 x) \\ = \quad & (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^1 x) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^2 x) \end{aligned} \tag{12}$$

From (11) and (12), it holds that

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \geq \mu_{\tilde{G}_i}^{-1}(h_i) \\ \geq \quad & (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^1 x) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^2 x). \end{aligned}$$

Therefore, MINMAX1($\hat{\mu}$) can be reduced to the following minmax problem.

[MINMAX2($\hat{\mu}$)]

$$\min_{x \in X, \lambda \in \Lambda} \lambda \tag{13}$$

subject to

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \geq (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^1 x) \\ & +T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda) \alpha_i^2 x), \\ & i = 1, \cdots, k \end{aligned} \tag{14}$$

It should be noted here that the constraints (14) can be reduced to a set of linear inequalities for some fixed value $\lambda \in \Lambda$. This means that an optimal solution (x^*, λ^*) of MINMAX2($\hat{\mu}$) is obtained by combined use of the bisection method with respect to $\lambda \in \Lambda$ and the first-phase of the two-phase simplex method of linear programming. The relationships between the optimal solution (x^*, λ^*) of MINMAX2($\hat{\mu}$) and D_p-Pareto optimal solutions can be characterized by the following theorem.

Theorem 1.

(1) If $x^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution of MINMAX2($\hat{\mu}$), then $x^* \in X, \hat{\mu}_i - \lambda^* \in H_i, i = 1, \cdots, k$ is a D_p-Pareto optimal solution.

(2) If $x^* \in X, h_i^* \in H_i, i = 1, \cdots, k$ is a D_p-Pareto optimal solution, then $x^* \in X$, $\lambda^* = \hat{\mu}_i - h_i^* = \hat{\mu}_i - \mu_{p_i}(p_i(x^*, h_i^*)), i = 1, \cdots, k$ is an optimal solution of MINMAX2($\hat{\mu}$) for some reference membership values $\hat{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$.

(Proof)

(1) From (14), it holds that $\hat{\mu}_i - \lambda^* \leq \mu_{p_i}(p_i(x^*, \hat{\mu}_i - \lambda^*)), i = 1, \cdots, k$. Assume that $x^* \in X, \hat{\mu}_i - \lambda^* \in H_i, i = 1, \cdots, k$ is not a D_p-Pareto optimal solution. Then, there exist $x \in X, h_i \in H_i, i = 1, \cdots, k$ such that

$$\begin{aligned} \mu_{D_{p_i}}(x, h_i) &= \min\{h_i, \mu_{p_i}(p_i(x, h_i))\} \\ &\geq \mu_{D_{p_i}}(x^*, \hat{\mu}_i - \lambda^*) \\ &= \hat{\mu}_i - \lambda^*, i = 1, \cdots, k, \end{aligned}$$

with strict inequality holding for at least one i. Then it holds that

$$h_i \geq \hat{\mu}_i - \lambda^*, i = 1, \cdots, k \tag{15}$$

$$\mu_{p_i}(p_i(x, h_i)) \geq \hat{\mu}_i - \lambda^*, i = 1, \cdots, k \tag{16}$$

From Assumption 1, (3) and $L^{-1}(h_i) \leq L^{-1}(\hat{\mu}_i - \lambda^*)$, (15) and (16) can be transformed as follows.

$$\begin{aligned} \mu_{\tilde{G}_i}^{-1}(h_i) &\leq \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*), i = 1, \cdots, k \\ \mu_{\tilde{G}_i}^{-1}(h_i) &\geq (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda^*) \alpha_i^1 x) \\ & \quad +T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda^*)) \\ & \quad \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda^*) \alpha_i^2 x), \\ & \quad i = 1, \cdots, k \end{aligned}$$

As a result, there exists $x \in X$ such that

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) - (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^1 x) \\ \geq \quad & T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda^*)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^2 x), \\ & i = 1, \cdots, k, \end{aligned}$$

which contradicts the fact that $x^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX2($\hat{\mu}$).
(2) Assume that $x^* \in X, \lambda^* \in \Lambda$ is not an optimal solution to MINMAX2($\hat{\mu}$) for any reference membership values $\hat{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$, which satisfy the equalities

$$\hat{\mu}_i - \lambda^* = h_i^* = \mu_{p_i}(p_i(x^*, h_i^*)), i = 1, \cdots, k. \quad (17)$$

Then, there exists some $x \in X, \lambda < \lambda^*$ such that

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) - (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x) \\ & \geq T_i^{-1}(\mu_{p_i}^{-1}(\hat{\mu}_i - \lambda)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x), \\ \Leftrightarrow \quad & \mu_{p_i}(p_i(x, \hat{\mu}_i - \lambda)) \geq \hat{\mu}_i - \lambda, i = 1, \cdots, k \quad (18) \end{aligned}$$

Because of (17),(18) and $\hat{\mu}_i - \lambda > \hat{\mu}_i - \lambda^*, i = 1, \cdots, k$, the following inequalities hold.

$$\mu_{p_i}(p_i(x, h_i)) \quad > \quad \mu_{p_i}(p_i(x^*, h_i^*)), i = 1, \cdots, k$$

where $h_i = \hat{\mu}_i - \lambda \in H_i$. Then, because of $h_i > h_i^*$, there exists $x \in X, h_i \in H_i, i = 1, \cdots, k$ such that

$$\mu_{D_{p_i}}(x, h_i) > \mu_{D_{p_i}}(x^*, h_i^*), i = 1, \cdots, k.$$

This contradicts the fact that $x^* \in X, h_i^* \in H_i, i = 1, \cdots, k$ is a D_p-Pareto optimal solution.

4 A FORMULATION THROUGH A FRACTILE OPTIMIZATION MODEL

If we adopt a fractile optimization model for the objective functions of MOSP, we can convert MOSP to the following multiobjective programming problem, where the decision maker specifies permissible probability levels $\hat{p}_i, i = 1, \cdots, k$ in his/her subjective manner (Sakawa et al., 2011).
[MOP6($\hat{p}$)]

$$\max_{x \in X, h_i \in [0,1], i = 1, \cdots, k} (h_1, \cdots, h_k) \quad (19)$$

subject to

$$p_i(x, h_i) \geq \hat{p}_i, i = 1, \cdots, k \quad (20)$$

where $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_k)$ is a vector of permissible probability levels. Since a distribution function $T_i(\cdot)$ is continuous and strictly increasing, the constraints (20) can be transformed to the following form.

$$\begin{aligned} & \hat{p}_i \leq p_i(x, h_i) \\ \Leftrightarrow \quad & \mu_{\tilde{G}_i}^{-1}(h_i) \geq (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) \\ & + T_i^{-1}(\hat{p}_i) \cdot (d_i^2 x - L^{-1}(h_i)\alpha_i^2 x) \quad (21) \end{aligned}$$

Let us define the right-hand side of the inequality (21) as follows.

$$\begin{aligned} f_i(x, h_i, \hat{p}_i) \quad \stackrel{\text{def}}{=} \quad & (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) \\ & + T_i^{-1}(\hat{p}_i) \cdot (d_i^2 x - L^{-1}(h_i)\alpha_i^2 x) \end{aligned} \quad (22)$$

Then, MOP6($\hat{p}$) can be equivalently transformed into the following form.
[MOP7($\hat{p}$)]

$$\max_{x \in X, h_i \in [0,1], i = 1, \cdots, k} (h_1, \cdots, h_k) \quad (23)$$

subject to

$$\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) \geq h_i, i = 1, \cdots, k \quad (24)$$

In MOP7($\hat{p}$), let us pay attention to the inequalities (24). $f_i(x, h_i, \hat{p}_i)$ is continuous and strictly increasing with respect to h_i for any $x \in X$. This means that the left-hand-side of (24) is continuous and strictly decreasing with respect to h_i for any $x \in X$. Since the right-hand-side of (24) is continuous and strictly increasing with respect to h_i, the inequalities (24) must always satisfy the active condition, that is, $\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) = h_i$, $i = 1, \cdots, k$ at the optimal solution. From such a point of view, MOP7($\hat{p}$) is equivalently expressed as the following form.
[MOP8($\hat{p}$)]

$$\max_{x \in X, h_i \in [0,1], i = 1, \cdots, k} \quad (\mu_{\tilde{G}_1}(f_1(x, h_1, \hat{p}_1)), \cdots, \mu_{\tilde{G}_k}(f_k(x, h_k, \hat{p}_k))) \quad (25)$$

subject to

$$\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) = h_i, i = 1, \cdots, k \quad (26)$$

In order to deal with MOP8($\hat{p}$), the decision maker must specify permissible probability levels $\hat{p}$ in advance. However, in general, the decision maker seems to prefer not only the larger value of a permissible probability level but also the larger value of the corresponding membership functions $\mu_{\tilde{G}_i}(\cdot)$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP8($\hat{p}$).
[MOP9]

$$\max_{x \in X, h_i \in [0,1], \hat{p}_i \in (0,1), i = 1, \cdots, k} (\mu_{\tilde{G}_1}(f_1(x, h_1, \hat{p}_1)), \cdots, \mu_{\tilde{G}_k}(f_k(x, h_k, \hat{p}_k)), \hat{p}_1, \cdots, \hat{p}_k)$$

subject to

$$\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) = h_i, i = 1, \cdots, k \quad (27)$$

It should be noted in MOP9 that permissible probability levels are not the fixed values but the decision variables.

Considering the imprecise nature of the decision maker's judgment, we assume that the decision maker has a fuzzy goal for each permissible probability level. Such a fuzzy goal can be quantified by eliciting the corresponding membership function. Let us denote a membership function of a permissible probability level $\hat{p}_i$ as $\mu_{\hat{p}_i}(\hat{p}_i)$. Then, MOP9 can be transformed as the following multiobjective programming problem.

[MOP10]

$$\max_{x\in X, h_i\in[0,1], \hat{p}_i\in(0,1), i=1,\cdots,k} (\mu_{\tilde{G}_1}(f_1(x,h_1,\hat{p}_1)), \cdots, \mu_{\tilde{G}_k}(f_k(x,h_k,\hat{p}_k)), \mu_{\hat{p}_1}(\hat{p}_1), \cdots, \mu_{\hat{p}_k}(\hat{p}_k))$$

subject to

$$\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) = h_i, i=1,\cdots,k \tag{28}$$

In order to elicit the membership functions appropriately, we suggest the following procedures. First of all, the decision maker sets the intervals $P_i = [p_{i\min}, p_{i\max}], i=1,\cdots,k$, where $p_{i\min}$ is an unacceptable maximum value of $\hat{p}_i$ and $p_{i\max}$ is a sufficiently satisfactory minimum value of $\hat{p}_i$. Throughout this section, we make the following assumption.

Assumption 2.
$\mu_{\hat{p}_i}(\hat{p}_i), i=1,\cdots,k$ are strictly increasing and continuous with respect to $\hat{p}_i \in P_i$, and $\mu_{\hat{p}_i}(p_{i\min}) = 0$, $\mu_{\hat{p}_i}(p_{i\max}) = 1$.

Corresponding to the interval P_i, the interval of h_i, which is defined as $H_i(P_i) = [h_{i\min}, h_{i\max}]$, can be obtained as follows. The maximum value $h_{i\max}$ can be obtained by solving the following problem.

$$\min_{x\in X, h_i\in[0,1]} f_i(x,h_i,p_{i\min}) \tag{29}$$

$$\text{subject to } \; h_i = \mu_{\tilde{G}_i}(f_i(x,h_i,p_{i\min})) \tag{30}$$

This is equivalent to the following problem.

$$h_{i\max} \stackrel{\text{def}}{=} \max_{x\in X, h_i\in[0,1]} h_i \tag{31}$$

subject to

$$\mu^{-1}_{\tilde{G}_i}(h_i) = (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) + T_i^{-1}(p_{i\min}) \cdot (d_i^2 x - L^{-1}(h_i)\alpha_i^2 x) \tag{32}$$

The optimal solution $x^*, h_i^*, i=1,\cdots,k$ of the above problem can be obtained by combined use of the bisection method with respect to $h_i \in [0,1]$ and the first-phase of the two-phase simplex method of linear programming. In order to obtain $h_{i\min}$, we first solve the following k linear programming problems.

$$\min_{x\in X, h_i\in[0,1]} f_i(x,h_i,p_{i\max}) \tag{33}$$

$$\text{subject to } \; h_i = \mu_{\tilde{G}_i}(f_i(x,h_i,p_{i\max})) \tag{34}$$

Let $(x_i^*, h_i^*), i=1,\cdots,k$ be the above optimal solution. Using the optimal solutions $(x_i^*, h_i^*), i=1,\cdots,k$, $h_{i\min}$ can be obtained as follows.

$$h_{i\min} \stackrel{\text{def}}{=} \min_{\ell=1,\cdots,k, \ell\neq i} \mu_{\tilde{G}_i}(f_i(x_\ell^*, h_\ell^*, p_{i\max})) \tag{35}$$

It should be noted here that, $\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i))$ is strictly decreasing with respect to $\hat{p}_i$. If the decision maker adopts the fuzzy decision (Sakawa, 1993) to integrate $\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i))$ and $\mu_{\hat{p}_i}(\hat{p}_i)$, MOP10 can be transformed into the following form.

[MOP11]

$$\max_{x\in X, \hat{p}_i\in P_i, h_i\in H_i(P_i), i=1,\cdots,k} \left(\mu_{D_{G_1}}(x,h_1,\hat{p}_1), \cdots, \mu_{D_{G_k}}(x,h_k,\hat{p}_k)\right) \tag{36}$$

subject to

$$\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) = h_i, i=1,\cdots,k \tag{37}$$

where

$$\mu_{D_{G_i}}(x,h_i,\hat{p}_i) \stackrel{\text{def}}{=} \min\{\mu_{\hat{p}_i}(\hat{p}_i), \mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i))\} \tag{38}$$

In order to deal with MOP11, we introduce a D_G-Pareto optimal solution concept.

Definition 2.
$x^* \in X, \hat{p}_i^* \in P_i, h_i^* \in H_i(P_i), i=1,\cdots,k$ is said to be a D_G-Pareto optimal solution to MOP11, if and only if there does not exist another $x \in X, \hat{p}_i \in P_i, h_i \in H_i(P_i), i=1,\cdots,k$ such that $\mu_{D_{G_i}}(x,h_i,\hat{p}_i) \geq \mu_{D_{G_i}}(x^*,h_i^*,\hat{p}_i^*), i=1,\cdots,k$ with strict inequality holding for at least one i, where $\mu_{\tilde{G}_i}(f_i(x^*,h_i^*,\hat{p}_i^*)) = h_i^*$, $\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) = h_i, i=1,\cdots,k$.

For generating a candidate of a satisfactory solution which is also D_G-Pareto optimal, the decision maker is asked to specify the reference membership values (Sakawa, 1993). Once the reference membership values $\hat{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ are specified, the corresponding D_G-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX3($\hat{\mu}$)]

$$\min_{x\in X, \hat{p}_i\in P_i, h_i\in H_i(P_i), i=1,\cdots,k, \lambda\in\Lambda} \lambda \tag{39}$$

subject to

$$\hat{\mu}_i - \mu_{\hat{p}_i}(\hat{p}_i) \leq \lambda, i=1,\cdots,k, \tag{40}$$

$$\hat{\mu}_i - h_i \leq \lambda, i=1,\cdots,k, \tag{41}$$

$$\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) = h_i, i=1,\cdots,k. \tag{42}$$

where

$$\Lambda = [\max_{i=1,\cdots,k} \hat{\mu}_i - 1, \min_{i=1,\cdots,k} \hat{\mu}_i]. \tag{43}$$

In the constraints (41) and (42), it holds that

$$
\begin{aligned}
& h_i = \mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) \geq \hat{\mu}_i - \lambda, \\
\Leftrightarrow\ & \mu_{\tilde{G}_i}^{-1}(h_i) = f_i(x,h_i,\hat{p}_i) \leq \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \\
\Leftrightarrow\ & \mu_{\tilde{G}_i}^{-1}(h_i) = (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) \\
& +T_i^{-1}(\hat{p}_i)\cdot(d_i^2 x - L^{-1}(h_i)\alpha_i^2 x) \\
& \leq \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda). \qquad (44)
\end{aligned}
$$

In the right hand side of (44), because of $L^{-1}(h_i) \leq L^{-1}(\hat{\mu}_i - \lambda)$ and $\alpha_i^1 x + T_i^{-1}(\hat{p}_i)\alpha_i^2 x > 0$, it holds that

$$
\begin{aligned}
& (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) \\
& +T_i^{-1}(\hat{p}_i)\cdot(d_i^2 x - L^{-1}(h_i)\alpha_i^2 x) \\
\geq\ & (d_i^1 x + T_i^{-1}(\hat{p}_i) d_i^2 x) \\
& -L^{-1}(\hat{\mu}_i - \lambda)\left(\alpha_i^1 x + T_i^{-1}(\hat{p}_i)\alpha_i^2 x\right). \qquad (45)
\end{aligned}
$$

Using (44) and (45), it holds that

$$
\begin{aligned}
& \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \\
\geq\ & (d_i^1 x + T_i^{-1}(\hat{p}_i) d_i^2 x) \\
& -L^{-1}(\hat{\mu}_i - \lambda)\left(\alpha_i^1 x + T_i^{-1}(\hat{p}_i)\alpha_i^2 x\right) \\
=\ & (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x) \\
& +T_i^{-1}(\hat{p}_i)\cdot(d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x). \qquad (46)
\end{aligned}
$$

Moreover, because of $\hat{p}_i \geq \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda)$, (46) can be transformed into the following form.

$$
\begin{aligned}
& T_i\left(\frac{\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) - (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x)}{d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x}\right) \\
& \geq \hat{p}_i \geq \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda), \\
\Leftrightarrow\ & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \geq (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x) \\
& +T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda))\cdot(d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x) \\
& \qquad (47)
\end{aligned}
$$

Therefore, MINMAX3($\hat{\mu}$) can be reduced to the following minmax problem.

[MINMAX4($\hat{\mu}$)]

$$
\min_{x \in X, \lambda \in \Lambda} \lambda \qquad (48)
$$

subject to

$$
\begin{aligned}
& \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) \geq (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x) \\
& +T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda))\cdot(d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x), \\
& i = 1,\cdots,k \qquad (49)
\end{aligned}
$$

It should be noted here that MINMAX4($\hat{\mu}$) is equivalent to MINMAX2($\hat{\mu}$). The relationships between the optimal solution (x^*,λ^*) of MINMAX4($\hat{\mu}$) and D_G-Pareto optimal solutions can be characterized by the following theorem.

Theorem 2.

(1) If $x^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution of MINMAX4($\hat{\mu}$), then $x^* \in X, \hat{p}_i^* = \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*) \in P_i, h_i^* = \hat{\mu}_i - \lambda^* \in H_i(P_i), i = 1,\cdots,k$ is a D_G-Pareto optimal solution.

(2) If $x^* \in X, \hat{p}_i^* \in P_i, h_i^* \in H_i(P_i), i = 1,\cdots,k$ is a D_G-Pareto optimal solution, then $x^* \in X$, $\lambda^* = \hat{\mu}_i - \mu_{\hat{p}_i}(\hat{p}_i^*) = \hat{\mu}_i - \mu_{\tilde{G}_i}(f_i(x^*,h_i^*,\hat{p}_i^*)), i = 1,\cdots,k$ is an optimal solution of MINMAX4($\hat{\mu}$) for some reference membership values $\hat{\mu} = (\hat{\mu}_1,\cdots,\hat{\mu}_k)$.

(Proof)

(1) From (49), it holds that

$$
\hat{\mu}_i - \lambda^* \leq \mu_{\tilde{G}_i}(f_i(x^*, \hat{\mu}_i - \lambda^*, \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))),
$$

and it is obvious that $\hat{\mu}_i - \lambda^* = \mu_{\hat{p}_i}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))$. Assume that $x^* \in X, \hat{\mu}_i - \lambda^* \in H_i(P_i), \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*) \in P_i, i = 1,\cdots,k$ is not a D_G-Pareto optimal solution. Then, there exist $x \in X, \hat{p}_i \in P_i, h_i \in H_i(P_i), i = 1,\cdots,k$ such that

$$
\begin{aligned}
\mu_{D_{G_i}}(x,h_i,\hat{p}_i) &= \min\{\mu_{\hat{p}_i}(\hat{p}_i), \mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i))\} \\
&\geq \mu_{D_{G_i}}(x^*, \hat{\mu}_i - \lambda^*, \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \\
&= \hat{\mu}_i - \lambda^*, i = 1,\cdots,k,
\end{aligned}
$$

with strict inequality holding for at least one i, and $\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) = h_i, i = 1,\cdots,k$. Then it holds that

$$
\begin{aligned}
\mu_{\hat{p}_i}(\hat{p}_i) &\geq \hat{\mu}_i - \lambda^*, i = 1,\cdots,k, \qquad (50) \\
\mu_{\tilde{G}_i}(f_i(x,h_i,\hat{p}_i)) &\geq \hat{\mu}_i - \lambda^*, i = 1,\cdots,k. \qquad (51)
\end{aligned}
$$

From Assumption 2 and (22), (50) and (51) can be transformed as follows.

$$
\begin{aligned}
\hat{p}_i &\geq \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*), i = 1,\cdots,k \\
\hat{p}_i &\leq T_i\left(\frac{\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) - (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x)}{d_i^2 x - L^{-1}(h_i)\alpha_i^2 x}\right), \\
& i = 1,\cdots,k
\end{aligned}
$$

Because of $L^{-1}(h_i) \leq L^{-1}(\hat{\mu}_i - \lambda^*), i = 1,\cdots,k$, there exists $x \in X$ such that

$$
\begin{aligned}
& \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) - (d_i^1 x - L^{-1}(h_i)\alpha_i^1 x) \\
& \geq T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))\cdot(d_i^2 x - L^{-1}(h_i)\alpha_i^2 x), \\
\Leftrightarrow\ & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) \geq \\
& (d_i^1 x + T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))\cdot d_i^2 x) \\
& -L^{-1}(h_i)(\alpha_i^1 x + T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))\cdot\alpha_i^2 x), \\
\Leftrightarrow\ & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) \geq (d_i^1 x + T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))\cdot d_i^2 x) \\
& -L^{-1}(\hat{\mu}_i - \lambda^*)(\alpha_i^1 x + T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*))\cdot\alpha_i^2 x) \\
& i = 1,\cdots,k.
\end{aligned}
$$

This contradicts the fact that $x^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX4($\hat{\mu}$).
(2) Assume that $x^* \in X, \lambda^* \in \Lambda$ is not an optimal solution to MINMAX4($\hat{\mu}$) for any reference membership values $\hat{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ which satisfy the equalities

$$\begin{aligned} \hat{\mu}_i - \lambda^* &= \mu_{\hat{p}_i}(\hat{p}_i^*) = \mu_{\tilde{G}_i}(f_i(x^*, h_i^*, \hat{p}_i^*)), \\ & \quad i = 1, \cdots, k. \end{aligned} \tag{52}$$

Then, there exists some $x \in X, \lambda < \lambda^*$ such that

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda) - (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^1 x) \\ & \geq T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda)\alpha_i^2 x), \\ \Leftrightarrow \quad & \mu_{\tilde{G}_i}(f_i(x, \hat{\mu}_i - \lambda, \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda)) \geq \hat{\mu}_i - \lambda \\ & i = 1, \cdots, k. \end{aligned} \tag{53}$$

Because of (52), (53) and $\hat{\mu}_i - \lambda > \hat{\mu}_i - \lambda^*, i = 1, \cdots, k$, the following inequalities hold.

$$\begin{aligned} \mu_{\hat{p}_i}(\hat{p}_i) &> \mu_{\hat{p}_i}(\hat{p}_i^*), i = 1, \cdots, k \\ \mu_{\tilde{G}_i}(f_i(x, \hat{h}_i, \hat{p}_i)) &> \mu_{\tilde{G}_i}(f_i(x^*, h_i^*, \hat{p}_i^*)), \\ & \quad i = 1, \cdots, k \end{aligned}$$

where $\hat{p}_i = \mu_{\hat{p}_i}^{-1}(\mu_i - \lambda) \in P_i$, $\hat{h}_i = \hat{\mu}_i - \lambda \in H_i(P_i), i = 1, \cdots, k$. This means that there exists some $x \in X, \hat{p}_i \in P_i, \hat{h}_i \in H_i(P_i), i = 1, \cdots, k$ such that $\mu_{D_{G_i}}(x, \hat{h}_i, \hat{p}_i) > \mu_{D_{G_i}}(x^*, h_i^*, \hat{p}_i^*), i = 1, \cdots, k$. This contradicts the fact that $x^* \in X, \hat{p}_i^* \in P_i, h_i^* \in H_i(P_i), i = 1, \cdots, k$ is a D_G-Pareto optimal solution.

5 AN INTERACTIVE ALGORITHM

In this section, we propose an interactive algorithm to obtain a satisfactory solution from among a D_G-Pareto optimal solution set. From Theorem 2, it is not guaranteed that the optimal solution (x^*, λ^*) of MINMAX4($\hat{\mu}$) is D_G-Pareto optimal, if it is not unique. In order to guarantee the D_G-Pareto optimality, we first assume that k constraints (49) of MINMAX4($\hat{\mu}$) are active at the optimal solution (x^*, λ^*), *i.e.*,

$$\begin{aligned} & \mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*) - (d_i^1 x^* - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^1 x^*) \\ = \quad & T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \\ & \cdot (d_i^2 x^* - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^2 x^*), \\ & i = 1, \cdots, k. \end{aligned} \tag{54}$$

If the j-th constraint of (49) is inactive, *i.e.*,

$$\begin{aligned} & \mu_{\tilde{G}_j}^{-1}(\hat{\mu}_j - \lambda^*) - (d_j^1 x^* - L^{-1}(\hat{\mu}_j - \lambda^*)\alpha_j^1 x^*) \\ & > T_j^{-1}(\mu_{\hat{p}_j}^{-1}(\hat{\mu}_j - \lambda^*)) \\ & \qquad \cdot (d_j^2 x^* - L^{-1}(\hat{\mu}_j - \lambda^*)\alpha_j^2 x^*), \\ \Leftrightarrow \quad & \mu_{\tilde{G}_j}^{-1}(\hat{\mu}_j - \lambda^*) > f_j(x^*, \hat{\mu}_j - \lambda^*, \mu_{\hat{p}_j}^{-1}(\hat{\mu}_j - \lambda^*)), \end{aligned} \tag{55}$$

we can convert the inactive constraint (55) into the active one by applying the bisection method for the reference membership value $\hat{\mu}_j \in [\lambda^*, \lambda^* + 1]$.

For the optimal solution (x^*, λ^*) of MINMAX4($\hat{\mu}$), where the active conditions (54) are satisfied, we solve the D_G-Pareto optimality test problem defined as follows.

[D_G-Pareto Optimality Test Problem.]

$$\max_{x \in X, \varepsilon_i \geq 0, i = 1, \cdots, k} w = \sum_{i=1}^{k} \varepsilon_i \tag{56}$$

subject to

$$\begin{aligned} & T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \cdot (d_i^2 x - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^2 x) \\ & + (d_i^1 x - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^1 x) + \varepsilon_i \\ = \quad & T_i^{-1}(\mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \cdot (d_i^2 x^* - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^2 x^*) \\ & + (d_i^1 x^* - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_i^1 x^*), i = 1, \cdots, k \end{aligned} \tag{57}$$

For the optimal solution of the above test problem, the following theorem holds.

Theorem 3.

For the optimal solution $\check{x}, \check{\varepsilon}_i, i = 1, \cdots, k$ of the test problem (56)-(57), if $w = 0$ (equivalently, $\check{\varepsilon}_i = 0, i = 1, \cdots, k$), $x^* \in X, \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i - \lambda^*) \in P_i, \hat{\mu}_i - \lambda^* \in H_i(P_i), i = 1, \cdots, k$ is a D_G-Pareto optimal solution.

Now, following the above discussions, we can present the interactive algorithm in order to derive a satisfactory solution from among a D_G-Pareto optimal solution set.

[An Interactive Algorithm.]

Step 1: The decision maker sets the membership functions $\mu_{\tilde{G}_i}(y), i = 1, \cdots, k$ for the fuzzy goals of the objective functions in MOFRLP.

Step 2: The decision maker sets his/her membership function $\mu_{\hat{p}_i}(\hat{p}_i)$.

Step 3: Set the initial reference membership values as $\hat{\mu}_i = 1, i = 1, \cdots, k$.

Step 4: Solve MINMAX4($\hat{\mu}$) by combined use of the bisection method $\lambda \in \Lambda$ and the first-phase of the two-phase simplex method of linear programming, and obtain the optimal solution (x^*, λ^*). For the optimal solution (x^*, λ^*), The corresponding D_G-Pareto optimality test problem (56)-(57) is formulated and solved.

Step 5: If the decision maker is satisfied with the current values of the D_G-Pareto optimal solution $\mu_{D_{G_i}}(x^*, h_i^*, \hat{p}_i^*), i = 1, \cdots, k$ where $\hat{p}_i^* = \mu_{\hat{p}_i}^{-1}(\hat{\mu}_i -$

Table 1: The parameters for LR-type fuzzy random variables $\tilde{\bar{c}}_{ij}$.

j	1	2	3	j	1	2	3
d^1_{1j}	2	1	3	d^2_{1j}	1.3	1.1	1.2
d^1_{2j}	-7	-7	-9	d^2_{2j}	1.1	1.2	1.1
α^1_{1j}	0.5	0.4	0.5	α^2_{1j}	0.05	0.04	0.05
α^1_{2j}	0.3	0.5	0.4	α^2_{2j}	0.05	0.04	0.05
β^1_{1j}	0.6	0.5	0.6	β^2_{1j}	0.06	0.05	0.06
β^1_{2j}	0.4	0.5	0.5	β^2_{2j}	0.06	0.06	0.05

$\lambda^*), h^*_i = \hat{\mu}_i - \lambda^*, i = 1, \cdots, k$, then stop. Otherwise, the decision maker updates his/her reference membership values $\hat{\mu}_i, i = 1, \cdots, k$, and return to Step 4.

6 A NUMERICAL EXAMPLE

We consider the following two-objective fuzzy random linear programming problem to demonstrate the feasibility of the proposed method under the hypothetical decision maker.

[MOFRLP]

$$\min_{x \in X} \quad \tilde{\bar{c}}_1 x = \tilde{\bar{c}}_{11} x_1 + \tilde{\bar{c}}_{12} x_2 + \tilde{\bar{c}}_{13} x_3$$
$$\min_{x \in X} \quad \tilde{\bar{c}}_2 x = \tilde{\bar{c}}_{21} x_1 + \tilde{\bar{c}}_{22} x_2 + \tilde{\bar{c}}_{23} x_3$$

where $X = \{(x_1, x_2, x_3) \geq 0 \mid 2x_1 + 6x_2 + 3x_3 \leq 150, 6x_1 + 3x_2 + 5x_3 \leq 175, 5x_1 + 4x_2 + 2x_3 \leq 160, 2x_1 + 2x_2 + 3x_3 \geq 90\}$, and it is assumed that a realization $\tilde{\bar{c}}_{ij}(\omega)$ of an LR-type fuzzy random variable $\tilde{\bar{c}}_{ij}$ is an LR fuzzy number whose membership function is defined as follows.

$$\mu_{\tilde{\bar{c}}_{ij}(\omega)}(s) = \begin{cases} L\left(\frac{d^1_{ij} + \bar{t}_i(\omega) d^2_{ij} - s}{\alpha^1_{ij} + \bar{t}_i(\omega)\alpha^2_{ij}}\right) & (s \leq \bar{d}_{ij}(\omega)), \\ R\left(\frac{s - d^1_{ij} + \bar{t}_i(\omega) d^2_{ij}}{\beta^1_{ij} + \bar{t}_i(\omega)\beta^2_{ij}}\right) & (s > \bar{d}_{ij}(\omega)), \end{cases}$$

where $L(t) = R(t) = \max\{0, 1-t\}$, and the parameters d^1_{ij}, d^2_{ij}, $\alpha^1_{ij}, \alpha^2_{ij}, \beta^1_{ij}, \beta^2_{ij}$ are given in Table 1.

Moreover, $\bar{t}_i, i = 1, 2$ are Gaussian random variables defined as $\bar{t}_i \sim \mathrm{N}(0, 1)$.

In MOFRLP, let us assume that the hypothetical decision maker sets the membership functions as follows (Step 1, 2).

$$\mu_{\tilde{G}_1}(f_1(x, h_1, \hat{p}_1)) = \frac{96.42857 - f_1(x, h_1, \hat{p}_1)}{96.42857 - 75}$$
$$\mu_{\tilde{G}_2}(f_2(x, h_2, \hat{p}_2)) = \frac{(-285) - f_2(x, h_2, \hat{p}_2)}{(-285) - (-332.143)}$$

$$\mu_{\hat{p}_1}(\hat{p}_1) = \frac{\hat{p}_1 - 0.401066}{(0.714968 - 0.401066)}$$
$$\mu_{\hat{p}_2}(\hat{p}_2) = \frac{\hat{p}_2 - 0.213304}{(0.812859 - 0.213304)}$$

Set the initial reference membership values as $(\hat{\mu}_1, \hat{\mu}_2) = (1, 1)$ (Step 3), and solve MINMAX4($\hat{\mu}$) by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming to obtain the corresponding D_G-Pareto optimal solution (x^*, λ^*) (Step 4).

$$\mu_{\tilde{G}_1}(f_1(x^*, h^*_1, \hat{p}^*_1)) = \mu_{\hat{p}_1}(\hat{p}^*_1) = 0.564271$$
$$\mu_{\tilde{G}_2}(f_2(x^*, h^*_2, \hat{p}^*_2)) = \mu_{\hat{p}_2}(\hat{p}^*_2) = 0.564271$$

The hypothetical decision maker is not satisfied with the current value of the D_G-Pareto optimal solution (x^*, λ^*), and, in order to improve $\mu_{D_{G_2}}(\cdot) = \min\{\mu_{\tilde{G}_2}(\cdot), \mu_{\hat{p}_2}(\cdot)\}$ at the expense of $\mu_{D_{G_1}}(\cdot) = \min\{\mu_{\tilde{G}_1}(\cdot), \mu_{\hat{p}_1}(\cdot)\}$, he/she updates his/her reference membership values as $(\hat{\mu}_1, \hat{\mu}_2) = (0.5, 0.6)$ (Step 5). Then, the corresponding D_G-Pareto optimal solution is obtained by solving MINMAX4($\hat{\mu}$) (Step 4).

$$\mu_{\tilde{G}_1}(f_1(x^*, h^*_1, \hat{p}^*_1)) = \mu_{\hat{p}_1}(\hat{p}^*_1) = 0.514421$$
$$\mu_{\tilde{G}_2}(f_2(x^*, h^*_2, \hat{p}^*_2)) = \mu_{\hat{p}_2}(\hat{p}^*_2) = 0.614421$$

For the current value of the D_G-Pareto optimal solution, the hypothetical decision maker updates his/her reference membership values $(\hat{\mu}_1, \hat{\mu}_2) = (0.52, 0.59)$ in order to improve $\mu_{D_{G_1}}(\cdot)$ at the expense of $\mu_{D_{G_2}}(\cdot)$ slightly (Step 5). The corresponding D_G-Pareto optimal solution is obtained by solving MINMAX4($\hat{\mu}$) (Step 4).

$$\mu_{\tilde{G}_1}(f_1(x^*, h^*_1, \hat{p}^*_1)) = \mu_{\hat{p}_1}(\hat{p}^*_1) = 0.529412$$
$$\mu_{\tilde{G}_2}(f_2(x^*, h^*_2, \hat{p}^*_2)) = \mu_{\hat{p}_2}(\hat{p}^*_2) = 0.599412$$

Then, since the hypothetical decision maker is satisfied with the current value of the D_G-Pareto optimal solution, stop the interactive processes (Step 5). The interactive processes under the hypothetical decision maker are summarized in Table 2.

In order to compare our proposed approach with the previous ones, let us obtain one of the Pareto optimal solutions of MOP8($\hat{p}$), which is defined in membership space, *i.e.*, $\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)), i = 1, \cdots, k$. Similar to MINMAX3($\hat{\mu}$), we can formulate the following minmax problem to obtain the Pareto optimal solution of MOP8($\hat{p}$).

[MINMAX5($\hat{p}, \hat{\mu}$)]

$$\min_{x \in X, h_i \in [0,1], i=1,\cdots,k, \lambda \in \Lambda} \lambda$$

subject to

$$\hat{\mu}_i - \mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) \leq \lambda, i = 1, \cdots, k,$$
$$\mu_{\tilde{G}_i}(f_i(x, h_i, \hat{p}_i)) = h_i, i = 1, \cdots, k.$$

Table 2: Interactive processes.

@	1	2	3
$\hat{\mu}_1$	1	0.5	0.52
$\hat{\mu}_2$	1	0.6	0.59
$\mu_{D_{G_1}}(x^*, h_1^*, \hat{p}_1^*)$	0.564271	0.514421	0.529412
$\mu_{D_{G_2}}(x^*, h_2^*, \hat{p}_2^*)$	0.564271	0.614421	0.599412
$\hat{p}_1^*$	0.578193	0.562545	0.567250
$\hat{p}_2^*$	0.551616	0.581684	0.572685
$f_1(x^*, h_1^*, \hat{p}_1^*)$	84.3370	85.4053	85.0840
$f_2(x^*, h_2^*, \hat{p}_2^*)$	-311.601	-313.966	-313.258

In MINMAX5$(\hat{p}, \hat{\mu})$, it is assumed that the decision maker sets his/her permissible probability levels as $\hat{p}_1 = \hat{p}_2 = 0.75$, and the reference membership values as $\hat{\mu}_1 = \hat{\mu}_2 = 1$. Then, the corresponding Pareto optimal solution is obtained as $f_1(x^*, h_1^*, 0.75) = 94.0338$, $f_2(x^*, h_2^*, 0.75) = -290.269$, $\mu_{\tilde{G}_i}(f_i(x^*, h_i^*, 0.75)) = 0.11176, i = 1, 2$. In our proposed algorithm, by solving MINMAX4$(\hat{\mu})$ for the reference membership values $\hat{\mu}_1 = \hat{\mu}_2 = 1$, the D_G-Pareto optimal solution is obtained as $f_1(x^*, h_1^*, \hat{p}_1^*) = 84.3370$, $f_2(x^*, h_2^*, \hat{p}_2^*) = -311.601$, $\hat{p}_1^* = 0.578193, \hat{p}_2^* = 0.551616$ (see the first iteration of Table 2). This means that a proper balance between permissible probability levels and the corresponding objective functions in a fractile optimization model is attained in membership space.

7 CONCLUSIONS

In this paper, we have proposed an interactive fuzzy decision making method for multiobjective fuzzy random linear programming problems to obtain a satisfactory solution from among a Pareto optimal solution set. In the proposed method, the decision maker is required to specify the membership functions for the fuzzy goals of not only objective functions but also the permissible probability levels. Pareto optimal concepts called D_p-Pareto optimal and D_G-Pareto optimal are introduced. The satisfactory solution can be obtained by updating the reference membership values and solving the corresponding minmax problem based on the linear programming technique. At the optimal solution of MINMAX2$(\hat{\mu})$ or MINMAX4$(\hat{\mu})$, it is expected that a proper balance between permissible possibility levels for a probability maximization model and permissible probability levels for a fractile optimization model is attained. In general, in order to deal with MOFRLP, the decision maker must specify many parameters in advance. Fuzzy operators such as the fuzzy decision will lighten his/her burden to specify such parameters as fixed values.

REFERENCES

Birge, J. R. and Louveaux, F. (1997). *Introduction to Stochastic Programming*. Springer.

Charnes, A. and Cooper, W. W. (1959). Chance constrained programming. *Management Science*, 6(1):73–79.

Dantzig, G. B. (1955). Linear programming under uncertainty. *Management Science*, 1(3-4):197–206.

Dubois, D. and Prade, H. (1980). *Fuzzy Sets and Systems: Theory and Applications*. Academic Press.

Kall, P. and Mayer, J. (2005). *Stochastic Linear Programming Models, Theory, and Computation*. Springer.

Katagiri, H., Ishii, H., and Itoh, T. (1997). Fuzzy random linear programming problem. In *Proceedings of Second European Workshop on Fuzzy Decision Analysis and Neural Networks for Management, Planning and Optimization*. Dortmund.

Kwakernaak, H. (1978). Fuzzy random variable-1. *Information Sciences*, 15:1–29.

Lai, V. and Hwang, C. (1992). *Fuzzy Mathematical Programming*. Springer.

Luhandjula, M. and Gupta, M. (1996). On fuzzy stochastic optimization. *Fuzzy Sets and Systems*, 81:47–55.

Puri, M. and Ralescu, D. (1986). Fuzzy random variables. *Journal of Mathematical Analysis and Applications*, 14:409–422.

Sakawa, M. (1993). *Fuzzy Sets and Interactive Multiobjective Optimization*. Plenum Press.

Sakawa, M., Nishizaki, I., and Katagiri, H. (2011). *Fuzzy Stochastic Multiobjective Programming*. Springer.

Wang, G.-Y. and Qiao, Z. (1993). Linear programming with fuzzy random variable coefficients. *Fuzzy Sets and Systems*, 57:295–311.

Yano, H. and Matsui, K. (2011). Fuzzy approaches for multiobjective fuzzy random linear programming problems through a probability maximization model. In *Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2011*, pages 1349–1354, Hong Kong.

Zimmermann, H.-J. (2011). *Fuzzy Sets, Decision-Making and Expert Systems*. Kluwer Academic Publishers, Boston.