ON MAXIMAL ROBUSTLY POSITIVELY INVARIANT SETS

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Abstract: This paper addresses the problem of computing maximal robustly positively invariant sets for discrete-time linear time-invariant systems with disturbance inputs. It is assumed that the disturbance is unknown, additive, but bounded. The main contribution is the determination of bound of the number of steps in the iterative construction of the maximal invariant sets.

1 INTRODUCTION

Set invariance plays a fundamental role in the analysis and design of control systems for constrained systems, since if the initial state is contained inside an invariant set, all future states will stay within the set and hence will satisfy the imposed system constraints, (Blanchini, 1999).

In literature, two types of convex sets are essentially used as candidate invariant sets: ellipsoidal and polyhedral sets. The use of ellipsoidal sets has the advantage that the complexity is fixed, (Kurzhanski and Varaiya, 2000), (Kurzhanski and Varaiya, 2002). However, they have a rather restricted shape, which may be very conservative in typical problems.

In this paper we will focus only on polyhedral sets in conjunction with linear dynamics.

The construction of maximal robustly positively invariant set for linear time-invariant (LTI) systems was studied in literature in different contexts, see for example the study in (Kolmanovsky and Gilbert, 1998). The method, proposed in this early studies constructs an invariant set by iteratively adding additional constraints until invariance is obtained. However, the iterative number is unknown in advance and can be very large.

In this paper we provide a novel method for constructing maximal robustly positively invariant sets for LTI systems that does not suffer from these drawbacks. Based on forward reachable sets, the method provides additional insight for a better understanding of the properties of the maximal robustly positively invariant sets. We will also discuss a method for computing an a priori lower bound relevant to the proposed method.

From literature, only the work in (Rakovic et al., 2004) proposed a method for determining an upper bound of the number of steps in the iterative construction of the maximal invariant sets. The method presented in the current paper offers a slight improvement for this upper bound.

The following notation will be used throughout the paper. $N \triangleq \{0,1,2,\ldots\}$ denotes the set of non-negative integers, N^+ denotes the set $N \setminus 0$ and $N_s \triangleq \{0,1,2,\ldots,s-1\}$. Whenever time is unspecified, a variable x stands for $x(k)$ for some $k \in N$.

For some $\varepsilon > 0$ we denote $B_p^n(\varepsilon) = \{x \in R^n : \|x\|_p \leq \varepsilon\}$, where $\|x\|_p$ is the $p-$norm of the vector $x = [x_1\ x_2 \ldots x_n]^T$, i.e. $\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$.

Given two sets $X_1 \subset R^n$ and $X_2 \subset R^n$, the Minkowski sum of the sets X_1 and X_2 is defined by $X_1 \oplus X_2 \triangleq \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2\}$. The Pontryagin difference of the set X_1 with respect to X_2 is defined by $X_1 \ominus X_2 = \{x |\ x + x_2 \in X_1,$ for all $x_2 \in X_2\}$.

The set X_1 is a proper subset of the set X_2 if and only if X_1 lies strictly inside X_2.

A C-set is a convex and compact set containing the origin as an interior point.

A polyhedron, or a polyhedral set, is the intersection of a finite number of half spaces. A polytope is a closed and bounded polyhedral set.

The paper is organized as follows. Section 2 deals with a general framework of robustly positively invariant sets. Section 3 is concerned with the minimal robustly positively invariant set while Section 4 is concerned with the maximal robustly constraint-admissible set. Section 5 is dedicated to the problem of computing an a priori lower bound. The simulation

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results are evaluated in Section 6 before drawing the conclusions.

2 ROBUSTLY POSITIVELY INVARIANT SET

Consider the following discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + \omega(k) \tag{1}$$

where $x(k) \in R^n$ and $\omega(k) \in R^n$.

The state is subject to the following polytopic constraint:

$$x \in X \tag{2}$$

where $X = \{x | H_x x \leq K_x\}$ is a C-set.

We assume that the disturbance sequence ω satisfies the constraint:

$$\omega \in W \tag{3}$$

where $W = \{\omega | H_w \omega \leq K_w\}$ is a C-set.

Recall the following definitions from (Blanchini and Miani, 2008):

Definition 1 (RPI Set)**.** The set Ω is robustly positively invariant (RPI) for system (1) if and only if $Ax + \omega \in \Omega$ for all $\omega \in W$ and all $x \in \Omega$. Equivalently Ω is RPI if and only if $A\Omega \oplus W \subseteq \Omega$.

Definition 2 (mRPI)**.** The set F_∞ is minimal RPI (mRPI) if it is a RPI set and contained in any RPI set.

It is well known that if the matrix A is not strictly stable, then F_∞ is unbounded. Therefore, in the sequel, we consider only the case when A is strictly stable.

It is also known that, the mRPI set is unique, compact and - in the case when W contains the origin - contains the origin.

Definition 3 (MRPI)**.** The set O_∞ is maximal RPI (MRPI) if it is a RPI set and contains every RPI set under a set of constraints (2), (3).

If the MRPI set is non-empty, then it is unique. Furthermore if X is a C set then the MPRI set is also a C set.

The link between the mRPI set F_∞ and the MRPI set O_∞ is given by the following theorem ((Kolmanovsky and Gilbert, 1998)):

Theorem 1. The following statements are equivalent:

1. the MRPI set O_∞ is non-empty,
2. $F_\infty \subset X$,
3. $X \ominus F_\infty$ contains the origin, where $\ominus$ denotes the Pontryagin difference.

Proof. The proof is not reported here. The reader is referred to (Kolmanovsky and Gilbert, 1998) for more details. □

Definition 4 (RAS)**.** A set Ω is a robustly constraint-admissible set (RAS) for system (1) if and only if $A^k x + A^{k-1}\omega(0) + A^{k-2}\omega(1) + \ldots + \omega(k-1) \in X, \forall k \in N$ for all $\omega \in W$ and all $x \in \Omega$. Furthermore if Ω contains every robustly constraint-admissible set then Ω is a maximal robustly constraint-admissible set (MRAS).

Theorem 2. The set Ω is a MRAS for system (1) if and only if this set is a MRPI set.

Proof. If Ω is MRPI and contained in X, then $Ax + \omega \in \Omega \subseteq X$ for any $\omega \in W$ and $x \in \Omega$. Hence Ω is a robustly constraint-admissible set, so Ω is contained in a MRAS.

Conversely, Ω is a MRAS. One has $A^2\Omega \oplus AW \oplus W \subseteq X$ or $A(A\Omega \oplus W) \oplus W \subseteq X$ or $A\Omega_1 \oplus W \subseteq X$, where $\Omega_1 = A\Omega \oplus W$. That means Ω_1 is a RAS. Hence, $\Omega_1 \subseteq \Omega$ or in another words, Ω is robustly invariant set and contained in the MRPI set. □

From the above theorem, one can conclude that the problem of finding MRPI sets is equivalent to the problem of finding MRAS. Therefore, in the rest of the paper, we consider only the problem of finding the MRAS for a given linear dynamics.

3 MINIMAL ROBUSTLY POSITIVELY INVARIANT SET

This section addresses the problem of approximating a mRPI.

It can be shown that in (Rakovic et al., 2005) the mRPI set F_∞ is the limit set of all the possible trajectories of (1) and defined as:

$$F_\infty = \sum_{i=0}^{\infty} A^i W$$

Since F_∞ is a Minkowski sum of infinitely many terms, its exact computation can be assured only under restrictive assumptions of nilpotent system dynamics, (Mayne and Schroeder, 1997).

Recall the following definition:

Definition 5 (mRPI $\varepsilon-$approximation)**.** Given a scalar $\varepsilon > 0$ and a set $\Omega \subset R^n$, the set $\Phi \subset R^n$ is an outer $\varepsilon-$approximation of Ω if

$$\Omega \subseteq \Phi \subseteq \Omega \oplus B_p^n(\varepsilon) \tag{4}$$

and an inner $\varepsilon-$approximation of Ω if

$$\Phi \subseteq \Omega \subseteq \Phi \oplus B_p^n(\varepsilon) \tag{5}$$

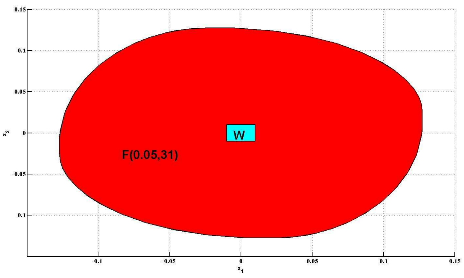

Figure 1: Approximation of F_∞ for example 1.

Denote

$$F_k = \sum_{i=0}^{k-1} A^i W$$

Theorem 3. If the set W contains the origin in its interior, then there exists a finite integer $r \in N^+$ and a scalar $\varepsilon \in (0,1]$ that satisfies:

$$A^r W \subseteq \varepsilon W \tag{6}$$

If (6) is satisfied, then

$$F(\varepsilon, r) = (1-\varepsilon)^{-1} F_r \tag{7}$$

is a convex, compact, RPI set of (1). Furthermore $F(\varepsilon, r)$ and $F_\infty \subset F(\varepsilon, r)$.

Proof. The proof is omitted here. The reader is referred to (Rakovic et al., 2005) for more details on this topic. □

4 MAXIMAL ROBUSTLY CONSTRAINT-ADMISSIBLE SET

In this section we consider the problem of the exact computation of the MRAS and start with the assumption that the mRPI set F_∞ is a proper subset of X.

Remark 1. The assumption $F_\infty \subset X$ is uncheckable but practically realistic by the fact that once we have an outer approximation, we can verify its inclusion in X.

Define the set $\Omega(s)$ by:

$$\Omega(s) = \left\{ x \;\middle|\; \begin{array}{cc} \{x\} & \subseteq X \\ \{Ax\} \oplus W & \subseteq X \\ \cdots & \\ \{A^{s-1}x\} \oplus \bigoplus_{k=0}^{s-2} A^k W & \subseteq X \end{array} \right\} \tag{8}$$

Theorem 4. There exists an index s that satisfies:

$$A^s X \oplus A^{s-1} W \oplus A^{s-2} W \oplus \ldots \oplus W \subseteq X \tag{9}$$

and the set $\Omega(s)$ defined in (8) is a MRAS for system (1).

Proof. One has

$$A^s X \oplus \bigoplus_{k=0}^{s-1} A^k W \subseteq A^s X \oplus \bigoplus_{k=0}^{\infty} A^k W \subseteq A^s X \oplus F_\infty \tag{10}$$

The fact that A is strictly stable and F_∞ is a proper subset of X confirm the existence of the index s by the fact that there will always an integer which makes $A^s X$ arbitrarily small.

For the second part of theorem, if $t \in N_s = \{0, 1, \ldots, s-1\}$, by the definition of the set $\Omega(s)$, for any $x \in \Omega(s)$ and any $w(k) \in W$ for $k = 0, 1, \ldots, t-1$ one has

$$A^t x \oplus \bigoplus_{k=0}^{t-1} A^k w(k) \in X \tag{11}$$

If $t \in N$ and $t \geq s$, it is possible to find a pair $p \in N$, $p \geq 1$ and $q \in N_s = \{0, 1, \ldots, s-1\}$ such that $t = ps + q$. Denote $\Psi = A^t \Omega(s) \oplus \bigoplus_{k=0}^{t-1} A^k W$, it follows that:

$$\begin{aligned} \Psi &= A^{ps+q}\Omega(s) \oplus \bigoplus_{k=0}^{ps+q-1} A^k W \\ &= A^{ps}\{A^q \Omega(s) \oplus \bigoplus_{k=0}^{q-1} A^k W\} \oplus \bigoplus_{k=0}^{ps-1} A^k W \\ &\subseteq A^{ps} X \oplus \bigoplus_{k=0}^{ps-1} A^k W \\ &= A^{(p-1)s}\{A^s X \oplus \bigoplus_{k=0}^{s-1} A^k W\} \oplus \bigoplus_{k=0}^{(p-1)s-1} A^k W \\ &\subseteq A^{(p-1)s} X \oplus \bigoplus_{k=0}^{(p-1)s-1} A^k W \\ &\ldots \\ &= A^s X \oplus \bigoplus_{k=0}^{s-1} A^k W \\ &\subseteq X \end{aligned}$$

Thus, for every $t \in N$, one has $A^t \Omega(s) \oplus \bigoplus_{k=0}^{t-1} A^k W \subseteq X$, hence $\Omega(s)$ is a constraint-admissible set. The fact that $\Omega(s)$ is a MRAS follows from the construction of this set. □

Clearly, if Ψ is any RPI set such that $F_\infty \subset \Psi \subset X$ and $A^s X \oplus \Psi \subset X$, then the set $\Omega(s)$ is a MRAS. This set Ψ can be obtained upon ultimate bounds in the case when A has real eigenvalues, for example using the results provided in the next theorem.

Theorem 5. (Kofman et al., 2007) Consider the system (1), let $A = TJT^{-1}$ be the Jordan decomposition of A and consider a bounding box for the set W. If this bounding box is described by the vector $\bar{\omega}$ which satisfies $|\omega| \leq \bar{\omega}$, $\forall \omega \in W$ then the set:

$$\Psi = \{x \left| \left|T^{-1}x\right| \leq (I - |J|)^{-1} \left|T^{-1}\right| \bar{\omega}\right.\} \tag{12}$$

is RPI, and thus contains F_∞.

Remark 2: Note that for any s_1 and s_2 that verify (9) one has $\Omega(s_1) = \Omega(s_2)$. One would like to find the smallest value of s such that (9) holds in order to reduce the number of redundant inequalities.

It is clear that, the set $\Omega(s)$ can be determined as follows:

$$\Omega(s) = \left\{ x \left| \begin{pmatrix} H_x \\ H_x A \\ \vdots \\ H_x A^{s-1} \end{pmatrix} x \leq K_x - K_s \right.\right\} \tag{13}$$

where K_s is a solution of the following s linear programs

$$K_s = \max_{\omega(0),\ldots,\omega(s-1)} \begin{pmatrix} 0 & 0 & \ldots & 0 \\ H_x & 0 & \ldots & 0 \\ H_x A & H_x & \ldots & 0 \\ & \ldots & \ldots & \\ H_x A^{s-2} & H_x A^{s-3} & \ldots & H_x \end{pmatrix} \begin{pmatrix} \omega(0) \\ \omega(1) \\ \vdots \\ \omega(s-1) \end{pmatrix}$$

subject to

$$\omega(k) \in W, \text{ k } = 0, 2, \ldots, s-1$$

It is worth noticing that the set $\Omega(s) = \{x|Hx \leq K\}$ may contain redundant inequalities. One can use the algorithm in (Kerrigan, 2000) to eliminate these inequalities.

5 A PRIORI LOWER BOUND COMPUTATION

In this section we will consider the problem of finding the smallest value of s such that the condition (9) holds.

5.1 The Theoretical Principle

One has

$$\begin{aligned} F_\infty &= \bigoplus_{k=0}^{\infty} A^k W = \bigoplus_{k=0}^{s-1} A^k W \oplus \bigoplus_{k=s}^{\infty} A^k W \\ &= \bigoplus_{k=0}^{s-1} A^k W \oplus A^s F_\infty \end{aligned}$$

then

$$\begin{aligned} X \ominus F_\infty &= X \ominus (\bigoplus_{k=0}^{s-1} A^k W \oplus A^s F_\infty) \\ &\supseteq (X \ominus \bigoplus_{k=0}^{s-1} A^k W) \ominus A^s F_\infty \\ &\supseteq A^s X \ominus A^s F_\infty \\ &\supseteq A^s (X \ominus F_\infty) \end{aligned} \tag{14}$$

Let $X_1 = X \ominus F_\infty = \{x|H_x^1 x \leq K_x^1\}$, it follows that $A^s X_1 \subseteq X_1$, so our problem is reduced to find the index s such that $A^s X_1 \subseteq X_1$.

Remark 3: Indeed, we obtain only bounds and not the exact index due to the fact that Pontryagin difference and Minkowski addition are not commutative operations.

Remark 4: Using the result in (Rakovic et al., 2004) an alternative upper bound r is obtained by exploiting the following set inclusion:

$$A^r X \subseteq X \ominus F_\infty$$

It is clear that the bound in (14) represents an improvement with respect to the result in (Rakovic et al., 2004) by the fact that $X \ominus F_\infty \subseteq X$.

5.2 Numerical Construction

Let $p_l(k)$ and $p_r(k)$ be solutions of following $2n$ linear programs:

$$\begin{aligned} p_l(k) &= \min x_k \\ \text{s.t. } & H_x^1 x \leq K_x^1, \end{aligned} \tag{15}$$

and

$$\begin{aligned} p_r(k) &= \min -x_k \\ \text{s.t. } & H_x^1 x \leq K_x^1, \end{aligned} \tag{16}$$

Define matrices R_{out} and R_{in} as follows:

$$R_{out} = \begin{pmatrix} R_o(1) & 0 & \ldots & 0 \\ 0 & R_o(2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_o(n) \end{pmatrix} \tag{17}$$

where $R_o(k) = \max(\left|p^l(k)\right|, |p^r(k)|)$, $k = 1, 2, \ldots, n$, and

$$R_{in} = \begin{pmatrix} R_i(1) & 0 & \ldots & 0 \\ 0 & R_i(2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_i(n) \end{pmatrix} \tag{18}$$

where $R_i(k) = \min(\left|p^l(k)\right|, |p^r(k)|)$, $k = 1, 2, \ldots, n$.

A set Φ_{out} defined as

$$\Phi_{out} = \{x \in R^n | x = R_{out} d, \|d\|_\infty \leq 1\} \tag{19}$$

is the smallest orthotope that contains X_1.

And a set Φ_{in} defined as

$$\Phi_{in}^1 = \{x \in R^n | x = R_{in}^1 d, \|d\|_\infty \leq 1\} \tag{20}$$

is the biggest orthotope that is contained in X_1.

In the case, when matrix A is not diagonizable, one can use the following algorithm to find the smallest index s such that $A^s X_1 \subseteq X_1$.

Consider the case when matrix A is diagonizable with $A = TJT^{-1}$, where T is a nonsingular matrix, J

Algorithm 1: Computation of the smallest index. The case when matrix A is not diagonizable.

Input: X_1, A
Output: s_o
2. Set $s = 1$;
3. **if** $A^s X_1 \subseteq X_1$ **then**
   Set $s_o = s$ and stop
else
   Continue
end
4. Set $s = s+1$ and go to step 3.

is a diagonal matrix of the eigenvalues of A and the spectral radius $\rho(A) \in (0,1)$. It is clear that if $A^s x \subseteq \Phi_{in}$ for any $x \in \Phi_{out}$ then $A^s x \in X_1$ for any $x \in X_1$. It follows that

$$\begin{aligned}
& A^s\Phi_{out} \subseteq \Phi_{in}^1 \\
& \Rightarrow A^s R_{out} d \subseteq \Phi_{in}^1, \ \|d\|_\infty \leq 1 \\
& \Rightarrow |A^s|_1 \leq \alpha, \alpha = \min \frac{R_{in}^1(i,i)}{R_{out}(i,i)}, \text{ i}=1,2,\ldots,n \\
& \Rightarrow |T|_1 \left|T^{-1}\right|_1 \rho^s \leq \alpha \\
& \Rightarrow s \geq \frac{\ln(\alpha)-\ln(|T|_1 \left|T^{-1}\right|_1)}{\ln(\rho)}
\end{aligned}$$

Denoting $\lceil s \rceil$ the smallest integer greater or equal to s, the set inclusion $A^s X_1 \in X_1$ is satisfied for every s such that $s \geq s^*$, where:

$$s^* = \left\lceil \frac{\ln(\alpha) - \ln(|T|_1 \left|T^{-1}\right|_1)}{\ln(\rho)} \right\rceil \qquad (21)$$

It is clear that this s^* may be not the smallest integer such that $A^s X \oplus \bigoplus_{i=0}^{s-1} A^i W \subseteq X$ holds. To the best of our knowledge, there is no effective method to determine analytically such s. One may use a bisection method for computing the smallest s, as follows:

Algorithm 2: Computation of the smallest index.

Input: s^*, X, W, A
Output: s_o
2. Set $s_1 = 0, s_2 = s^*$;
3. Set $s = \lceil \frac{s_1+s_2}{2} \rceil$;
4. **if** $A^s X \oplus \bigoplus_{k=1}^{s-1} A^k W \subseteq X$ **then**
   set $s_2 = s$
else
   $s_1 = s$
end
5. **if** $s_2 - s_1 = 1$ **then**
   set $s_o = s_2$ and stop
else
   go to step 3
end

Remark 5. The condition $A^s X \oplus \bigoplus_{k=1}^{s-1} A^k W \subseteq X$ can be verified by solving the following linear programs:

$$\begin{aligned}
& J = \max\{H_x A^s x + H_x A_{s-1}\omega(0) + \ldots + H_x \omega(s-1)\} \\
& \text{s.t. } x \in X \\
& \qquad \omega(i) \in W, \text{ i } = 1,2,\ldots,s-1
\end{aligned}$$

and after that checking condition $J \leq K_x$.

6 EXAMPLES

To show the effectiveness of the proposed method, two examples will be considered in this section. For both of these examples, to solve linear programs, we used the Multi-parametric toolbox, (Kvasnica et al., 2004).

6.1 Example 1

This example is taken from (Rakovic and Fiacchini, 2008). Consider the following discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + \omega(k) \qquad (22)$$

where

$$A = 0.9 \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} 0.8916 & 0.1225 \\ -0.1225 & 0.8916 \end{pmatrix}$$

with $\theta = \frac{\pi}{3}$ and

$$X = \{x \in R^2 |\, \|x\|_\infty \leq 100\} \cap \{x \in R^2 | x_2 \geq -20\}. \qquad (23)$$

The disturbance set is

$$W = \{\omega \in R^2 |\, \|\omega\|_\infty \leq 0.01\} \qquad (24)$$

Figure 1 presents the disturbance set W and the RPI set obtained by using theorem 3.

Using algorithm 1, one obtains $s_o = 19$.

Figure 2 shows the maximal robustly positively invariant set O_∞.

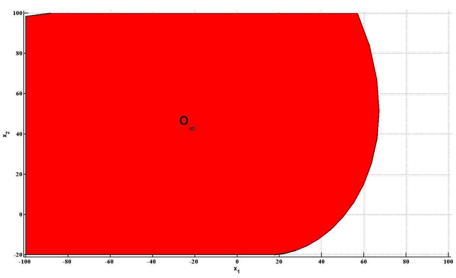

Figure 2: the MRPI set O_∞ for example 1.

6.2 Example 2

To show the ability of the algorithm to cope efficiently with a higher order systems, we will use a 4th order system in this example.

Consider the following discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + \omega(k) \qquad (25)$$

where

$$A = \begin{pmatrix} 0.5042 & 0.0618 & 0.6935 & 0.1406 \\ 0.3070 & 0.1811 & 0.4636 & -0.0106 \\ -0.4748 & -0.0911 & 0.1162 & 0.1502 \\ 0.1940 & 0.0771 & 0.6828 & 0.3539 \end{pmatrix}$$

and

$$X = \{x \in R^4 | \ \|x\|_\infty \leq 50\} \cap \{x \in R^4 | \ \left|\sum_{i=1}^{4} x_i\right| \leq 10\}$$

The disturbance set is

$$W = \{\omega \in R^4 | \ \|\omega\|_\infty \leq 0.1\}$$

Using theorem 3, Figure 3 illustrates the disturbance set W and the RPI set with $\varepsilon = 0.32$ and $r = 4$.

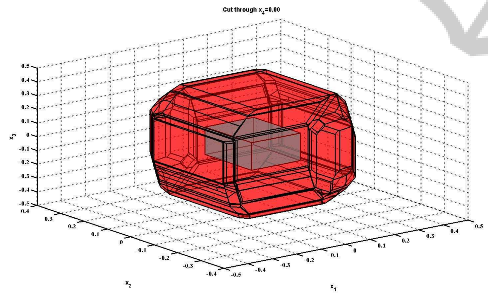

Figure 3: Approximation of F_∞ for example 2, cut through $x_4 = 0$.

Using algorithm 1, one obtains $s_o = 7$.

Figure 4 illustrates the maximal robustly positively invariant set O_∞.

7 CONCLUSIONS

This paper discussed the characterization of the maximal robustly positively invariant sets for discrete-time linear time-invariant systems with disturbance inputs by providing upper bounds for the iterative construction.

It was shown that the maximal robustly positively invariant set and the maximal robustly constraint-admissible set are the same. Examples of a second order plant, and a fourth order plant are given.

The simulation results show the effectiveness of the proposed methods.

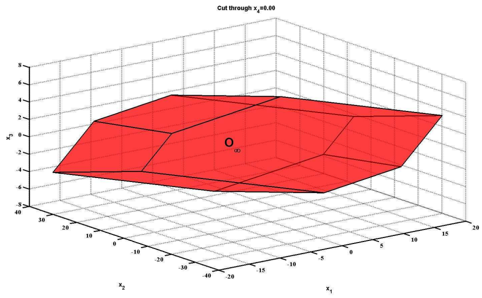

Figure 4: The maximal robustly positively invariant set for example 2, cut through $x_4 = 0$.

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