RELATIONSHIP BETWEEN LEVY DISTRIBUTION AND TSALLIS DISTRIBUTION

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Abstract: This paper describes the relationship between a stable process, the Levy distribution, and the Tsallis distribution. These two distributions are often confused as different versions of each other, and are commonly used as mutators in evolutionary algorithms. This study shows that they are usually different, but are identical in special cases for both normal and Cauchy distributions. These two distributions can also be related to each other. With proper equations for two different settings (with Levy's kurtosis parameter α < 0.3490 and otherwise), the two distributions match well, particularly for $1 \leq \alpha \leq 2$.

1 INTRODUCTION

Researchers have conducted many studies on computational methods that are motivated by natural evolution [1-6]. These methods can be divided into three main groups: genetic algorithms (GAs), evolutionary programming (EP), and evolutionary strategies (ESs). All of these groups use various mutation methods to intelligently search the promising region in the solution domain. Based upon these mutation methods, researchers often use three types of mutation variate to produce random mutation: Gaussian, Cauchy and Levy variates. Gaussian and Cauchy variates are special cases of the Levy process. Lee et al. (Lee and Yao, 2004) introduced the Levy process, used Mantegna's algorithm (Mangetna, 1994) to produce the Levy variate, and showed that the algorithm is useful for Levy's kurtosis parameter $\alpha > 0.7$. Iwamatsu generated the Levy variate of the Levy-type distribution, which is just an approximation, using the algorithm proposed by Tsallis and Stariolo (Iwamatsu, 2002). Iwamatsu's contribution is the usage of Tsallis and Stariolo's algorithm to generate the Tsallis variate and apply it to the mutation in the evolutionary programming. The Tsallis variate is not the Levy stable process, but is very similar. The paper first introduces the stable process and Tsallis distribution. Equations show that these two distributions are generally different, but are identical for two special distributions, i.e. the normal and Cauchy distributions. This section also provides two equations to link the parameters in the Levy distribution and Tsallis distribution so that they can be approximated to each other. Various examples show that they are quite similar, but not identical.

The Levy stable process can not only be used in simulated annealing, evolutionary algorithms, as a model for many types of physical and economic systems, it also has quite amazing applications in science and nature. In the case of animal foraging, food search patterns can be quantitatively described as resembling the Levy process. For example, researchers have studied reindeer, wandering albatrosses, and bumblebees and found that their random walk resembles Levy flight behavior (see example in Viswanathan et al. (Viswanathan and Afanasyev, etal, 2000), Edwards et al. (Edwards and Philips et al, 2007)). The strength of Levy flight in animal foraging is obvious, as it helps foragers find food and survive in severe environments.

2 THEORETICAL DEPLOYMENT

In probability theory, a Lévy skew alpha-stable distribution or even just a stable distribution is a four parameter family of continuous probability distributions. The parameters are classified as location and scale parameters μ and c, and two shape parameters β and α, which roughly correspond to measures of skewness and kurtosis, respectively. The stable distribution has the important property of stability. Except for possibly different shift and scale

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parameters, a stochastic variable, which is a linear combination of several independent variables with stable distribution, has the same stable distribution.

The Lévy skew stable probability distribution is defined by the Fourier transform of its characteristic function $\varphi(\mathrm{t})$ (Voit, 2003)

$$f(x;\alpha,\beta,c,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)e^{-itx}dt \quad (1)$$

where $\varphi(\mathrm{t})$ is defined as:

$$\varphi(\mathrm{t}) = \exp\left(itu - |ct|^{\alpha}\left(1-i\beta\,\mathrm{sgn}(t)\Phi\right)\right) \quad (2)$$

where $\mathrm{sgn}(t)$ is just the sign of t, and Φ is given by

$$\Phi=\tan(\pi\alpha/2)$$

for all α except α = 1, in which case:

$$\Phi=-(2/\pi)\log|t|\,.$$

Note that the range of each parameter is the kurtosis $\alpha\in(0,2]$, the skewness $\beta\in[-1,1]$, the scale $c>0$, and the location $\mu\in(-\infty,\infty)$. Assuming that the distribution is symmetric $(\beta=0)$, the center of its location is zero $(\mu=0)$, then Eq. (2) can be simplified as

$$\varphi(\mathrm{t}) = \exp\left(-|ct|^{\alpha}\right). \quad (3)$$

Inserting Eq. (3) into (1) produces

$$f(x;\alpha,0,c,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-|ct|^{\alpha})e^{-itx}dt\,. \quad (4)$$

Let

$$\gamma=|c|^{\alpha} \quad (5)$$

and using the Euler formula

$$e^{i\theta}=\cos\theta+i\sin\theta \quad (6)$$

and considering only the real part of Eq. (6), it is easy to show that

$$f(x;\alpha,0,c,0)=\mathrm{L}_{\alpha,\gamma}(x)=\frac{1}{\pi}\int_{0}^{\infty}\exp(-\gamma t^{\alpha})\cos(tx)dt\,. \quad (7)$$

This equation is identical to $\mathrm{L}_{\alpha,\gamma}(y)$ in Lee's and Mantegan's paper, though the current study changes the variable y to x.

When α =2, the stable process in Eq. (3) becomes a normal distribution. Using the characteristic function of a normal distribution with a zero mean and a variance of σ_1^2 (Papoulis, 1990), which is $\varphi(t)=\exp(-\frac{\sigma_1^2t^2}{2})$, it is easy to show that the variance σ_1^2 of Eq. (3) is $2c^2$. As for the Cauchy distribution (α =1), its characteristics function is $\varphi(t)=\exp(-|ct|)$ and the corresponding probability density function is

$$g_2(x)=\frac{1}{c\pi}\frac{1}{1+(x/c)^2}\,. \quad (8)$$

The Tsallis distribution (Tsallis and Stariolo, 1996) in one dimension is written as follows

$$g(x;q,T)=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}\frac{T^{-1/(3-q)}}{\left\{1+(q-1)\frac{x^2}{T^{2/(3-q)}}\right\}^{1/(q-1)}}\,. \quad (9)$$

Note that the ranges of parameters q and T are $q\in[1,3)$ and $T>0$, respectively. The follow section investigates the relationship between the parameters α, c of $f(x;\alpha,0,c,0)$ in Eq. (1) and the q and T of $g(x;q,T)$ in Eq. (9).

According to Iwamatsu, when $q\to1^+$, the Tsallis distribution becomes a normal distribution

$$g_1(x;q\to1^+,T)=\frac{1}{\sqrt{\pi\sigma^2}}\exp(-(x/\sigma)^2)\,, \quad (10)$$

and when q =2, it becomes the Cauchy distribution

$$g_2(x;2,T)=\frac{1}{\pi\sigma}\cdot\frac{1}{1+(x/\sigma)^2}\,. \quad (11)$$

Note that $\sigma=T^{1/(3-q)}$ is a scale parameter, and is not the usual meaning of standard deviation in a normal distribution. The scale parameter σ is a function of q and T, and with different q it has different function forms of T. For example, if q=1, then $\sigma=\sqrt{T}$, whereas if q=2, then $\sigma=T$. The true standard deviation of the normal distribution in Eq. (10) is $\sigma_1=\sigma\sqrt{\frac{1}{2}}=\sqrt{\frac{T}{2}}$, which renders the standard form of normal distribution as

$$g_1(x;q\to 1^+,T)=\frac{1}{\sigma_1\sqrt{2\pi}}\exp(-\frac{1}{2}(x/\sigma_1)^2). \quad (12)$$

As indicated above, the variance of normal distribution as a special case of Levy distribution is $2c^2$ and the variance of normal distribution as a special case of Tsallis distribution is $\frac{T}{2}$. Therefore, if the two normal distributions are identical, the parameters between the Levy distribution and Tsallis distribution must satisfy the following constraint,

$$2c^2=\frac{T}{2}. \quad (13)$$

By the same token, apply the equality of the Cauchy distribution and compare Eq. (8) and (11). It is clear that

$$c=\sigma=T. \quad (14)$$

Equations (13) and (14) establish the link between parameter c of the Levy stable process in Eq. (7) and the parameters q and T of the Tsallis distribution in Eq. (9) for the special cases of normal (α=2, q=1) and Cauchy distributions (α=1, q=2). Since this is derived only from special cases of α=1 or 2, this study proposes a general model between parameters c and α in Eq. (7) and parameters q and T in Eq. (9) as follows

$$\alpha c=T^{1/(3-q)}. \quad (15)$$

This model establishes the first relationship between two sets of parameters (α,c) and (q,T). Note that when α =2 (which implies q =1), Eq. (15) reduces to Eq. (13), whereas when α =1 (which implies q =2), Eq. (15) reduces to Eq. (14). The second relationship between (α,γ) and (q,T) is inspired by Mantegna's equation,

$$L_{\alpha,1}(0)=\frac{\Gamma(1/\alpha)}{\pi\alpha} \quad (16)$$

which describes the probability density in Eq. (7) with scale parameter c=1 (implying γ=1, through Eq. (5)) at $x=0$. Recall that when $x=0$, the probability density for Tsallis distribution renders

$$g_q(x=0)=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}T^{-1/(3-q)}. \quad (17)$$

Combining Eq. (16) and (17) leads to

$$\frac{\Gamma(1/\alpha)}{\pi\alpha}=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}T^{-1/(3-q)}. \quad (18)$$

Equation (18) gives another constraint between parameter α in Eq. (7) and parameters q and T in Eq. (9) when γ=1. Since this equation (18) is derived from the special case of γ=1, this study proposes a general model between parameters α and γ in Eq. (7) and parameters q and T in Eq. (9) as follows

$$\frac{\Gamma(1/\alpha)}{\pi\alpha(\gamma^{1/\alpha})}=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}T^{-1/(3-q)}. \quad (19)$$

Note that when γ=1, Eq. (19) reduces to Eq. (18). Therefore, by combining Eq. (5), (15) and (19) and making some substitution in the parameters, this study obtains two equations to define the relationship between (α,γ) and (q,T) as

$$\alpha\gamma^{1/\alpha}=T^{1/(3-q)}$$

$$\frac{\Gamma(1/\alpha)}{\pi}=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}. \quad (20)$$

Substituting Eq. (5) into Eq. (20), the similar relationship between (α,c) and (q,T) leads to

$$\alpha c=T^{1/(3-q)}$$

$$\frac{\Gamma(1/\alpha)}{\pi}=\left(\frac{q-1}{\pi}\right)^{1/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}. \quad (21)$$

Equation (21) states that two constraints are required to establish the relationship between the Levy stable process parameters (α, c) and the Tsallis distribution parameters (q, T) so that the two distributions will be equal in the special cases of two categories. The first category includes the normal and Cauchy distributions, in which the Levy and Tsallis distributions are identical. In the second category, the scale parameter c =1, and the Levy and Tsallis distributions coincide only at the peak of the distribution. We do not know how close these two distribution match in other regions of the variate domain in the second category. To determine the relationship between these two X, apply equation (21) as follows. For the special case of normal equation（for stable process α =2 and for Tsallis distribution $q \to 1^+$） we obtain

$$\begin{gathered} 2c = T^{1/2} \\ \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \end{gathered} \qquad (22)$$

The first constraint in Eq. (22) states that $2c^2 = \frac{T}{2}$, which is exactly the same as Eq. (13). This shows that the two distributions are equal in the special case of a normal distribution. For the special case of a Cauchy equation（for stable process α =1 and for Tsallis distribution q =2）, Eq. (21) yields

$$\begin{gathered} c = T \\ \frac{1}{\pi} = \frac{1}{\pi} \end{gathered} \qquad (23)$$

The first constraint in Eq. (23) equals Eq. (14), which shows that the two distributions are identical in the special case of a Cauchy distribution. As above, Eq. (20) can be substituted for α =2 and 1 to obtain the following equations

$$\begin{gathered} 2\gamma^{1/2} = T^{1/2} \\ \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \end{gathered} \qquad (24)$$

and

$$\begin{gathered} \gamma = T \\ \frac{1}{\pi} = \frac{1}{\pi} \end{gathered} \qquad (25)$$

Equations (24) and (25) define the constraints between (α, γ) and (q, T) for normal and Cauchy distributions. Next, this study verifies a normal distribution case using a graph. Let α =2 and $c = 0.8$, and by Eq. (22) we have

$$T = 4c^2 = 2.56. \qquad (26)$$

The probability density of Eq. (7) can be calculated through numerical integration. Fortunately, John Nolan has developed a program, stable.exe, to perform the required calculations and made it available on his website. Using the stable.exe program from Nolan (Nolan, 1998) to evaluate the probability density function (pdf) of Eq., (7) with α =2 and $c = 0.8$, this study compares it with the Tsallis pdf of $q \to 1^+$ and T =2.56. Figure 1 shows the results of this graph comparison, indicating that the pdfs are identical. The blue line represents the stable.exe program and the green squares represent the Tsallis pdf. This study selects the S_0 stable process in the stable.exe and sets its gamma value at the $c = 0.8$ to calculate its probability density function. In other words, the gamma value in stable.exe is not γ but c in our definition on the stable process in Eq. (1) and (5).

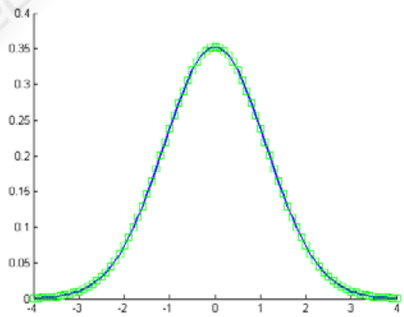


Figure 1: The comparison between Levy and Tasllis with α =2 and c =0.8.

This study also tests the Cauchy distribution with α =1 and c =0.75, and compares it with the Tsallis pdf with $q = 2$ and T =0.75. Figure 2 shows these results, which are clearly also identical.

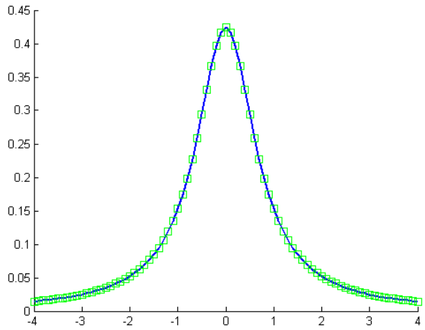


Figure 2: The comparison between Levy and Taslis with α =1 and c =0.75.

Comparing the general cases of α =2/3 and c =2.4 and substituting them into Eq. (21) leads to

$$1.6 = T^{1/(3-q)}$$
$$\frac{1}{2\sqrt{\pi}} = \left(\frac{q-1}{\pi}\right)^{1/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)} . \qquad (27)$$

The right hand side of the second constraint in Eq. (27) is clearly a monotonically decreasing function of q . Thus, the solution for q is uniquely determined. Solving q first in the lower part of Eq. (27) produces q =2.1263. Substituting this value into the upper part of Eq. (27) then leads to T =1.5078. Figure 3 compares the Levy distribution with parameters α =2/3 and c =2.4 and the Tsallis distribution with q =2.1263 and T =1.5078, showing that they are not identical. This is not a surprise because the Tsallis distribution is generally not a Levy stable process, and Levy stable processes usually do not have an analytical form except for special cases [7]. However, they are quite close. This means that the Tsallis distribution can be a good approximation of the Levy distribution. Using the values α =0.9 and c =1.4 produces similar results, and Figure 4 shows that they are almost identical.

A comparison of Figures 3 and 4 clearly shows that as α becomes smaller, the match between Levy and Tsallis decreases. Equation (21) helps explain the deviation between these two distributions. For the sake of clarity, repeat the second part of Eq. (21) in Eq. (28) as follows. Here y has two meanings: one is the function of q ($y(q)$); the other is the function of α $(y(\alpha))$.

$$\frac{\Gamma(1/\alpha)}{\pi} = y = \left(\frac{q-1}{\pi}\right)^{1/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)} \qquad (28)$$

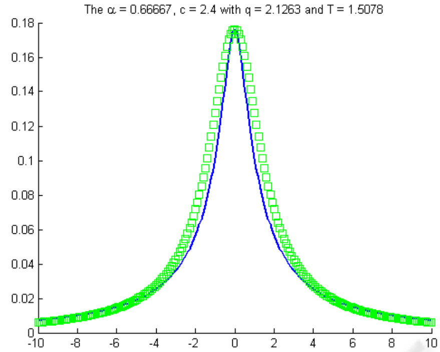


Figure 3: The comparison between Levy and Taslis with α =2/3 and c =2.4.

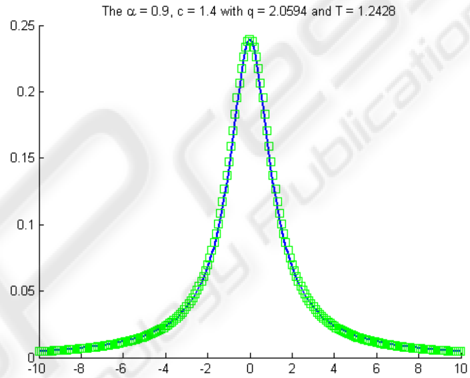


Figure 4: The comparison between Levy and Taslis with α =0.9 and c =1.4.

As mentioned above, the right hand side of the second equation in Eq. (21) is a monotonically decreasing function of q for $q \in (1,3]$. Figure 5 shows the results.

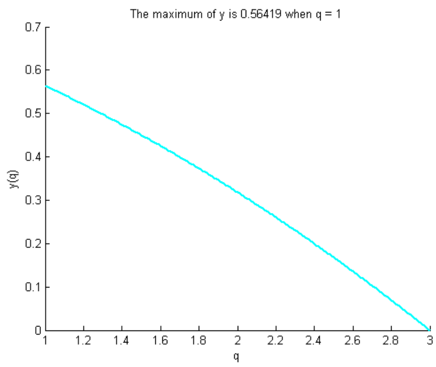


Figure 5: Function of $y(q)$.

The maximum of y is 0.56419 when $q \to 1^+$. Note also that the left hand side of Eq. (28) is a convex function of α . As $\alpha \to 0^+$, $y(\alpha)$ approaches to infinity and decreases to a minimum as α increases up to 0.684. Then $y(\alpha)$ climbs upwards and reaches its local extreme when α goes to 2. Figure 6 shows the results.

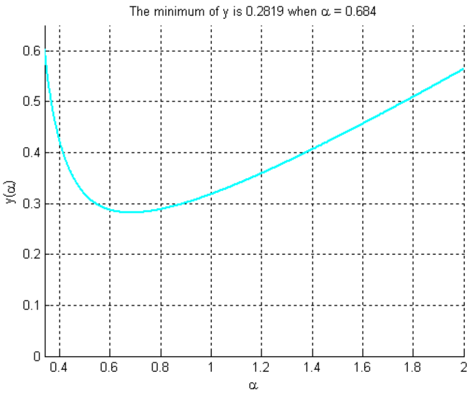


Figure 6: Function of $y(\alpha)$.

To produce a solution of q for a given α in Eq. (28), the value of $y(q)$ in Fig. 5 must equal that of $y(\alpha)$ in Fig. 6. Since the range of $y(q)$ is less than or equal to 0.56419 and the range of $y(\alpha)$ can go up to infinity, it is clear that for certain ranges of α, there is no solution for q that satisfies Eq. (28). This creates the first problem. On the other hand, since $y(\alpha)$ is a convex function with a minimum of 0.2819, thus for $y(q)$ less than 0.2819 (which implies that $q > 2.127$), there is no solution for α that satisfies Eq. (28). The third problem is related to the second problem. When $y(q) \in [0.2819, 0.56419]$, there are two solutions for α. This creates the possible dilemma that for a set of (q,T), there are two sets of (α,c) that satisfy Eq. (28). This means there is a possibility that (q,T) and (α,c) do not form a one-on-one mapping, which is an undesirable situation. The following section solves the third problem of finding proper (α,c) given a set of (q,T). The other two problems, are solved in a similar manner.

Two examples can be used to demonstrate the procedure of solving two solutions for α. The map between (α,c) and (q,T) can be unique even if there are two solutions of α for a given q in Eq. (28). Fix one α (where α <0.684) first, and then use Eq. (28) to determine its left hand side. Then solve another α (where α >0.684) by applying the left hand side of Eq. (28) again. This approach produces two values of α (say, α_1 and α_2) for a common $y(\alpha)$. Using the right hand side of Eq. (28), solve for a unique q. Further, assume a value of T such that there are two sets of (α,c), say (α_1, c_1) and (α_2, c_2), in Eq. (21) for a given set of (q,T). Which one of (α_1, c_1) and (α_2, c_2) is a better match to (q,T)? Numerical examples show that a significant difference may exist between (α_1, c_1) and (α_2, c_2) in the matter of resemblance to (q,T). Thus, find two sets of (α,c), which are (α_1, c_1) and (α_2, c_2), and compare them to the (q,T) to find a better match between (α,c) and (q,T). The following section provides two numerical examples. First let the first choice of α (where α >0.684) be α_1 =1, then the left hand side of Eq. (28) is $\frac{1}{\pi} = 0.3183$, and the other α (where α <0.684), which renders the same $y(\alpha)$, be α_2=0.5. In both cases, the corresponding q=2. Now further assume that T =1, and substitute α = α_1 =1, T =1, and q=2 into the upper part of Eq. (21). This produces $c = c_1$=1, which is a standard Cauchy distribution, and is the same result obtained above. The Levy distribution (with parameters α_1=1 and c_1=1) and Tsallis distribution (with parameters q =2 and T =1) coincide. On the other hand, substituting $\alpha = \alpha_2$=0.5, q=2, and T=1 leads to $c = c_2$=2. It is clear that the Levy distribution (with parameters α_2=0.5 and c_2=2) is not a Cauchy distribution, and therefore will not be equal to the Tsallis distribution (with parameters q=2 and T=1). Figure 7 shows the departure between α =0.5, c =2, and q =2, T=1. This figure shows that the departure can be quite large between (α,c) and (q,T) for an improper choice of (α,c). Thus, selecting the correct values of (α,c) for a given value of (q,T) is a crucial task: the right choice leads to an exact match, whereas the wrong choice produces an out of shape match.

Next, consider the first problem: how to find q when α < 0.3490? The solution in this study is based on Deng's paper [22], in which the relationship between α and q when $x \to \infty$ is

$$q = 1 + \frac{2}{1+\alpha}. \tag{29}$$

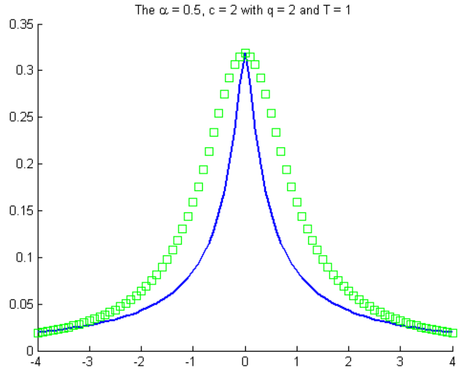


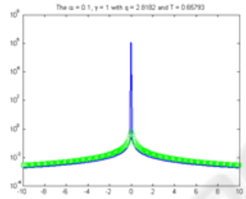
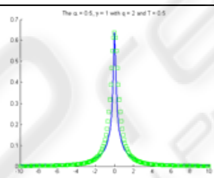
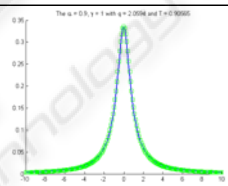
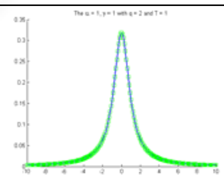
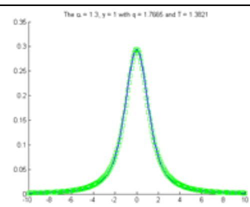
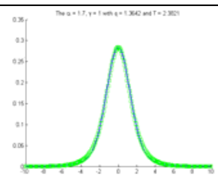
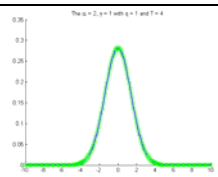
Figure 7: The comparison between α =0.5, c =2 (blue line), and q =2, T =1 (green squares).

Equation (29) clearly shows that as $\alpha \to 0^+$, $q \to 3^-$. Recall that the range for α is $(0,2]$ and the range for q is $[1,3)$. Therefore, Eq. (29) satisfies the one to one relationship between α and q when α < 0.3490. Note that Eq. (29) also solves the second problem, i.e., when q > 2.127 there is no solution of α in Eq. (28). Simply replacing Eq. (28) with Eq. (29) immediately solves the first and second problems. Combining Eq. (29) with the first part of Eq. (21) produces a new equation for solving (q,T) given a set value of (α,c) when α < 0.3490. This equation is

$$\alpha c = T^{1/(3-q)} \qquad q = 1 + \frac{2}{1+\alpha} . \tag{30}$$

The purpose for substituting Eq. (29) for Eq. (28) is to focus on the match between the two distributions in the heavy tails instead of on the peak of the distribution. This is because the heavy tails count more (or have more impact) when α < 0.3490. To show the effect of Eq. (21) and (30), try different α values with c = γ =1 using Eq. (21) and (30). Check the relationship between α and the match quality between the two distributions. Table 1 lists these results, showing that when $\alpha \geq 1$, the match quality between Levy and Tsallis distributions is either perfect or excellent. At $\alpha < 1$, the quality deteriorates a bit. When α < 0.3490, the two distributions match very well on the heavy tails except for the narrow region near the origin, where they are significantly different. Note that the blue line represents the Levy stable process, whereas the green squares stand for the Tsallis distribution. Note that for the case of α =0.1, the green squares rise above the blue line in the region from $-10 \leq x \leq 10$. If the domain of x is extended in absolute value to 10000, the two will match almost exactly on the heavy tails. This result is not shown here for the sake of brevity. The difference between

Table 1: Match quality vs various α .

α	Graph result	Match quality
0.1		fair
0.5		good
0.9		good
1.0		exact
1.3		excellent
1.7		excellent
2.0		exact

them is less than $1.5*10^{-7}$ in absolute value. Taking the Levy stable process as the standard and approximating it by the Tsallis distribution shows that the relative deviation is about 11.35%. The relative deviation decreases as the absolute value of x increases.

3 CONCLUSIONS

This study thoroughly investigates the relationship between the parameters (α, c) and (q, T) in the Levy distribution and the Tsallis distribution. Results show that they are usually totally different, except for two special cases of normal and Cauchy distributions. However, they can be approximated to each other through linking equation in (21) or (30) depending on whether or not the kurtosis parameter is α < 0.3490. When $\alpha \geq 1$, the match quality between the Levy and Tsallis distributions is either perfect or excellent. When α <1, the quality deteriorates a bit. When α <0.3490, except on the narrow region near origin where the two have a significant difference, the two match very well on the heavy tails.

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