Scoring Systems and Large Margin Perceptron Ranking using Positive Weights

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Abstract. Large Margin Perceptron learning with positive coefficients is proposed in the context of so-called scoring systems used for assessing creditworthiness as stipulated in the Basel II central banks capital accord of the G10-states. Thus a potential consistency problem can be avoided. The approximate solution of a related ranking problem using a modified large margin algorithm producing positive weights is described. Some experimental results obtained from a Java prototype are exhibited. An important parallelization using Java concurrent programming is sketched. Thus it becomes apparent that combining the large margin algorithm presented here with the pocket algorithm can provide an attractive alternative to the use of support vector machines. Related algorithms are briefly discussed.

1 Introduction

At least since the Basel II central banks capital accord of the G10-states, cf. e.g. [1], the individual objective rating of the credit worthiness of customers has become an important problem. To this end so-called scoring systems, cf. e.g. [12], [23], [17], [6] have been used for quite some time. Generally these systems are simple classifiers that are implemented as (linear) discriminants where customer characteristics such as income, property assets, liabilities and the likes are assigned points or grades and then a weighted average is computed, where a customer is judged "good" or "bad" according to whether the average exceeds a cut-off point or not. In an extreme case the attributes are just binary ones where 0 respectively 1 signifies that the property does not hold respectively holds. This situation frequently arises in practice. The weights can then either be computed using classical statistical methods or more recently employing artificial neural networks, cf. e.g. [19], provided that suitable bank records are available for training.
However, the use of only two classes for the classification of customers presents certain problems, see e.g. [1], [10]. Hence in this paper it is assumed that training data are available, where banking customers are divided into mutually disjoint risk classes C_1, C_2, …, C_k. Here class C_i is preferred to C_j if $i<j$. It was shown in [8] how this preference relation may be learned employing a generalized version of the Krauth/Mezard large margin perceptron algorithm to solve the associated ranking

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problem. Unfortunately a consistency problem arises in this context if the attributes are assigned points or grades that are not exclusively taken from the set {0, 1}. The solution to this problem can, however, be achieved if positive weights only are used. This restriction leads to an interesting modification of the large margin ranking algorithm given in [10], that will be presented here. Moreover a sketch solution for a parallel implementation using concurrent Java programming leading to better performance on multiprocessor PCs will be described.
Note that the use of several classes has been investigated beforehand, see e.g. [2], p. 237. Moreover, the use of ranking functions has been recognized in an information retrieval context, cf, e.g. [25], for solving certain financial problems, cf. [15], and for collaborative filtering, cf. [20], [21]. However, at least in the banking business, ranking functions, as described in section 2 below, see also [7], [22], apparently have not been used before for the rating of customers, cf. [12]. Note also that in none of these works the consistency problem alluded to above has been recognized.

2 Reduction of the Ranking Problem

Suitable anonymous training data from a large German bank were available. In abstract terms then t vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ from $\Re^n$ (think of these as having grades assigned to individual customer characteristics as their entries) together with their risk classification (i.e. their risk class C_s for $1 \le s \le k$, where the risk classes were assumed to constitute a partition of pattern space) were given. Hence implicitly a preference relation (partial order) "$\rangle$" in pattern space was determined for these vectors by

$$\mathbf{x}_i \rangle \mathbf{x}_j \quad \text{if } \mathbf{x}_i \in C_i \text{ and } \mathbf{x}_j \in C_j \text{ where } i < j.$$

It was then required to find a map $m_{\mathbf{w}}: \Re^n \to \Re$ that preserves this preference relation, where the index $\mathbf{w}$ of course denotes a weight vector. More precisely one must have

$$\mathbf{x}_i \rangle \mathbf{x}_j \Rightarrow m_{\mathbf{w}}(\mathbf{x}_i) > m_{\mathbf{w}}(\mathbf{x}_j)$$

If one now specializes by setting $m_{\mathbf{w}}(\mathbf{x}) := \langle \varphi(\mathbf{x}), \mathbf{w} \rangle$, denoting the scalar product by $\langle .,. \rangle$ and an embedding of $\mathbf{x}$ in a generally higher (m-) dimensional feature space by φ, then the problem reduces to finding a weight vector $\mathbf{w}$ and constants ("cut-offs") $c_1 > c_2 > \ldots > c_{k-1}$ such that

$$\mathbf{x} \in C_1 \text{ if } \langle \varphi(\mathbf{x}), \mathbf{w} \rangle > c_1$$
$$\mathbf{x} \in C_s \text{ if } c_{s-1} \ge \langle \varphi(\mathbf{x}), \mathbf{w} \rangle > c_s \quad \text{for} \quad s = 2, 3, \ldots, k-1$$
$$\mathbf{x} \in C_k \text{ if } c_{k-1} \ge \langle \varphi(\mathbf{x}), \mathbf{w} \rangle.$$

The problem may then be reduced further to a standard problem:
Let $\mathbf{e}_i$ denote the i-th unit vector in $\Re^{k-1}$ considered as row vector and construct a matrix $\mathbf{B}$ of dimension $(m_1+2m_2+k-2)\times(m+k-1)$, where $m_1 := |C_1 \cup C_k|$ (here $|S|$ denotes the cardinality of set S) and $m_2 := |C_2 \cup C_3 \ldots \cup C_{k-1}|$, as follows:

$\mathbf{B} := \begin{bmatrix} \mathbf{R} \\ \mathbf{D} \end{bmatrix}$, dimension R = (k-2) ×(m+k-1), and the i-th row of $\mathbf{R}$ is given by the row vector $(0, \ldots, 0, \mathbf{e}_i - \mathbf{e}_{i+1})$ with m leading zeros. Moreover $\mathbf{D}$ is described by:
For every vector $\mathbf{x}$ in C_1 respectively C_k $\mathbf{D}$ contains a row vector $(\varphi(\mathbf{x}), -\mathbf{e}_1)$ respectively $(-\varphi(\mathbf{x}), \mathbf{e}_{k-1})$, whilst for every vector $\mathbf{x}$ in C_s with $1 < s < k$ it contains the

vectors ($\boldsymbol{\varphi}(\mathbf{x})$, $-\mathbf{e}_s$) and ($-\boldsymbol{\varphi}(\mathbf{x})$, $\mathbf{e}_{s-1}$). The reduction of the problem to a system of inequalities is then proved by the following lemma.

Lemma 1: A weight vector $\mathbf{w}$ and constants $c_1 > c_2 > \ldots > c_{k-1}$ solving the ranking problem may (if they exist) be obtained by solving the standard system of inequalities $\mathbf{Bv} > \mathbf{0}$ where $\mathbf{v} := (\mathbf{w}, c_1, c_2, \ldots, c_{k-1})^T$.
Proof (see also [7]): Computation.
Of course, it must be admitted that the existence of a suitable weight vector $\mathbf{v}$ is by no means guaranteed. However, at least in theory, the map $\boldsymbol{\varphi}$ may be chosen such that the capacity of a suitable separating hyperplane is large enough for a solution to exist with high probability, cf. [4].
The price one has to pay for this increased separating capacity consists on the one hand of larger computation times. On the other hand, and perhaps more importantly, a loss of generalization capabilities due to a higher VC-dimension of the separating hyperplanes, cf. e.g. [24], must be taken into account. In order to improve the generalization properties here a large margin perceptron ranking algorithm producing positive weights based on the work of Krauth and Mezard will be presented. This may be used to construct a separating hyperplane that has the large margin property for the vectors correctly separated by the pocket algorithm. The reader should compare this to the large margin ranking described in [20]: There the problem is solved using a (soft margin) support vector machine. Unfortunately computation of the complete set of cut-offs requires the solution of an additional linear optimization problem. Moreover no positivity condition is mentioned.

3 Large Margin Ranking and the Positivity Condition

The work of Krauth and Mezard concerning large margin perceptron learning is described in [14]. Certain modifications were introduced in order to obtain a large margin ranking (LMR) algorithm, cf. [9]. Further modifications lead to large margin ranking observing the positivity condition derived from consistency considerations.

3.1 The Consistency Problem and the Positivity Restriction

In the experiments described below (using “real life data” obtained from a German bank) in the solution to the ranking problem invariably some of the weights computed were negative. However, it seems natural to rate a customer cust1 characterized by a vector $\mathbf{x}_1$ better than a customer cust2 characterized by a vector $\mathbf{x}_2$, provided that $\mathbf{x}_1 \geq \mathbf{x}_2$, where the inequality between vectors is supposed to hold if it holds for all corresponding entries. After all cust1 is then in all criteria rated at least as well as cust2.
If negative weights are admitted though the situation could arise where one customer cust1 is rated equal to another customer cust2 in all but one criteria, where he is rated better. In this case surely cust1’s overall rating should be better than cust2’s. If the corresponding weight happens to be negative, however, the contrary would be the case. Thus it definitely seems to be desirable to demand that weights should be

positive (zero might possibly still be allowed). Hence a suitably modified large margin ranking algorithm is presented below.

3.2 Pseudo Code for LMR with Positive Weights

The **pseudo code** for this algorithm reads as follows.

Input. Binary vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_t$ (or vectors with integer entries) from Z^n with corresponding classifications $b_1, b_2, ..., b_t$ from {1, 2, …, k} (where the classes $C_1, C_2, …, C_k$ for simplicity have been denoted by their indices) as training vectors, and a function $\varphi: Z^n \rightarrow Z^m$, where in general m>n. In addition a real number $\alpha > 0$ must be chosen.

Output. A weight vector **w** having positive entries only and k-1 cut-offs c_i satisfying $c_1 > c_2 > … > c_{k-1}$ as vector **c** that approximate the maximal margin solution of the ranking problem. The approximation improves with increasing α (under certain conditions, see below).

Initialize **w**, **c** with **0**, **0**.
Loop

For given $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), ..., \varphi(\mathbf{x}_t)$ compute the minimumm of the following expressions:

(i) $\langle\varphi(\mathbf{x}_i), \mathbf{w}\rangle - c_s$ if $1 \le s \le k\text{-}1$ for $\mathbf{x}_i \in C_s$, $1 \le i \le t$

(ii) $-\langle\varphi(\mathbf{x}_i), \mathbf{w}\rangle + c_{s-1}$ if $2 \le s \le k$

(iii) $\langle\mathbf{w}, \mathbf{e}_i\rangle$ if this expression is ≤ 0.

Then m either has the form

(a) $m = \langle\varphi(\mathbf{x}_j), \mathbf{w}\rangle - c_s$ for some j und $\mathbf{x}_j \in C_s$ or

(b) $m = -\langle\varphi(\mathbf{x}_k), \mathbf{w}\rangle + c_{s-1}$ for some k und $\mathbf{x}_k \in C_s$ or

(c) $m = \langle\mathbf{w}, \mathbf{e}_i\rangle$ for some i.

If $m > \alpha$ then {display **w**, **c;** stop;}
Else

If (a) then {$\mathbf{w} := \mathbf{w} + \varphi(\mathbf{x}_j)$; $c_s := c_s - 1$;} End If
If (b) then {$\mathbf{w} := \mathbf{w} - \varphi(\mathbf{x}_k)$; $c_{s-1} := c_{s-1} + 1$;} End If
If (c) then {$\mathbf{w}_i := \mathbf{w}_i + 1$;} End If

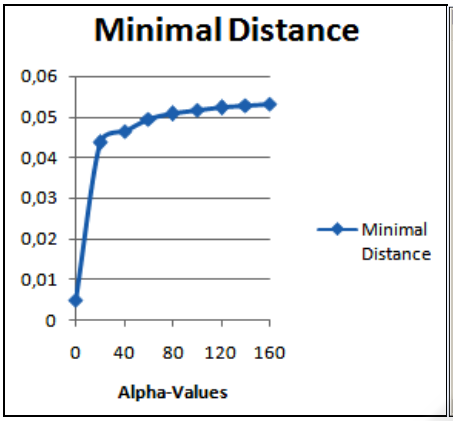
End If

The proof that the algorithm converges to the maximal margin of separation under a certain additional condition (which, as is frequent in perceptron learning, cf. [3], [16], [18], is hard to verify but does not seem very restrictive in practice) is provided in the appendix.

3.3 Experimental Results

In order to test the large margin algorithm with positive weights and with a view to further extensions a Java prototype was constructed. This was connected to an Access database via an JDBC:ODBC bridge.

The experiments were carried out with 58 data vectors, which allowed perfect separation, provided by a German financial institution. The customers had been divided into 5 preference classes (for a detailed description of the data see [10]). The experiments were conducted on a standard laptop (3.2 GHz clock, 2 GB RAM). For simplicity the function φ appearing in the algorithm was taken to be the identity.
As a measure of the quality of approximation the distance of the "worst-classified" element to the nearest cut-off was computed. In order to obtain a reference value it should be mentioned that the optimal margin for arbitrary weights was obtained as 0.0739745 by quadratic programming in [10]. It should also be pointed out that as for arbitrary weights, cf. [10], the time requirements increased roughly linearly with α, the maximum time taken (for α = 160) being 29656 msecs. The results obtained are displayed in diagram 1.



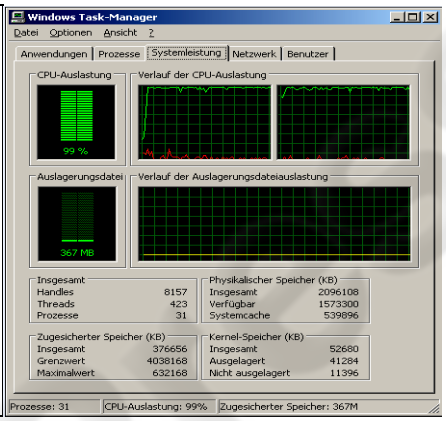


Diagram1 **Diagram 2**

As may be seen from diagram 1 the quality of approximation to the optimal solution improves quite fast with increasing α up to about 40. Thereafter, however, only slow progress is made. Nevertheless, for practical purposes this approximation may be quite acceptable. In view of the additional condition needed to guarantee convergence it should also be pointed out that in all other experiments, see also 3.4 below, the algorithm gave very good results by increasing the distance of the worst classified element from the nearest cutoff by a factor of about 4 to 10.

3.4 Implementation with Concurrent Java Programming

In view of the fact that multi-processor machines are beginning to dominate the PC market it seems important that the algorithm presented can be parallelized by exploiting concurrent Java programming. This is essentially achieved by splitting the main loop of the algorithm into n (= number of processors) loops of roughly equal size and assigning them to separate Java worker threads. In each thread then the required minima are computed separately. The results are passed to the main thread which computes the overall minimum and updates the weights and cutoffs accordingly. Thereafter the updated values are passed to the worker threads and the

process is repeated. The required synchronization can be obtained by using the Java classes “BlockingQueue” and “CountDownLatch”, cf.
http://java.sun.com/javase/6/docs/api/java/util/concurrent/BlockingQueue.html
and for the latter class
http://java.sun.com/javase/6/docs/api/java/util/concurrent/CountDownLatch.html.
The authors of the present paper conducted a preliminary test with a dual core machine that showed that the Java virtual machine splits the work between the two processors rather nicely if the loop is sufficiently large, see diagram 2 above.
In fact with roughly 12 000 data sets (divided into two preference classes with binary attribute values only; sufficiently many data encompassing more preference classes could unfortunately not be obtained) the speed up compared with the ordinary program version apeared to be close to 100% so that very little administrative overhead was incurred (incidentally: The total CPU time turned out to be in the order of three minutes which is surprisingly small). However, a test with a small number of data even lead to an increase in CPU time, since evidently the administrative overhead became too large. Nevertheless in practical situations often large data sets prevail. In view of this fact and because a realization of the dual (kernel) version would definitely lead to large CPU times (preliminary tests indicate CPU times of several hours even for quadratic polynomial kernels), the parallelization should be extremely helpful. The precise variation of the required overhead with the number of processors is still the subject of ongoing research. Further details will be reported elsewhere.

4 Conclusion and Outlook

A new large margin ranking algorithm producing positive weights only and thus removing a potential consistency problem has been presented. Encouraging experimental evidence has been obtained using “real life” data from a financial institution. In contrast to the wide margin ranking algorithm described in [20] it can be implemented with a surprisingly compact Java encoding. This is due to the fact that it can be seen as an extension of classical perceptron learning. Moreover the algorithm allows easy parallelization.
On the other hand, of course, it gives only an approximate solution and also needs an additional condition to guarantee convergence, which may, however, as indicated by the experimental results, be quite satisfactory for practical applications. In addition the algorithm works for separable sets only. However, it is intended to combine it with a modified version of the pocket algorithm, cf.[11], by applying it to those data sets only that are correctly separated. This seems attractive since that way certain approximations inherent to the soft margin support vector machine as utilized in [20] are avoided.
Finally a few comments on related algorithms seem in order. The large margin algorithm in [20] has been briefly mentioned already. The ranking algorithms in [5] and [13] appear inferior from the results given in [20]. In [26] large margin perceptron learning was introduced for the pocket algorithm. However, in spite of reasonable experimental evidence, the theoretical basis appears slightly shaky, for details see e.g. the conclusion in [9]. The ranking algorithm in [22] (soft margin

version) appears to contain a gap since the monotonicity condition for the cut-offs seems to be neglected. Moreover an additional vector is ignored without explaining the consequences. In short then the algorithm closest to the one presented here seems to appear in [20]. Of course, it has been tested in a completely different context and an objective comparison concerning the banking application envisaged here is still outstanding. Moreover none of these algorithms appear to deal with the potetntial consistency problem discussed in 3.2.

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Appendix

Convergence Theorem

For simplicity here the following problem is considered (the convergence theorem for the algorithm given in 3.4 may then be deduced by applying the results of section 2):
Given sample vectors (patterns) $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p \in Z^n$ find a vector $\mathbf{w}^*$ having positive entries only such that. $\langle \mathbf{w}^*, \mathbf{x}_j \rangle \geq \alpha$ for j= 1,2, …, p and a fixed constant $\alpha > 0$.
Assume throughout that such a vector exists with $\|\mathbf{w}^*\| = \alpha / \Delta_{opt}$, where Δ_{opt} denotes the maximal separation of the samples from zero.
Then consider the following algorithm
Set $\mathbf{w}_0 := 0$ and

$$\mathbf{w}_{t+1} := \mathbf{w}_t + \mathbf{x}_{j(t)} \quad \text{if } \langle \mathbf{w}_t, \mathbf{x}_{j(t)} \rangle < \alpha,$$

where $\mathbf{x}_{j(t)}$ is defined by

$$\mathbf{x}_{j(t)} := \begin{cases} \min\{\min_{1\leq k\leq p} \langle \mathbf{w}_t, \mathbf{x}_k \rangle, \min_{1\leq l\leq n} \langle \mathbf{w}_t, \mathbf{e}_l \rangle\} & \text{if } \min_{1\leq l\leq n} \langle \mathbf{w}_t, \mathbf{e}_l \rangle \leq 0 \\ \min_{1\leq k\leq p} \langle \mathbf{w}_t, \mathbf{x}_k \rangle, \text{ otherwise.} \end{cases}$$

General Convergence Properties

After M updates according to this rule suppose that pattern $\mathbf{x}_j$ has been visited m_j times and that the j-th unit vector $\mathbf{e}_j$ has been visited λ_j times where

$$\sum_{j=1}^{p} M_j = M_1 \text{ and } \Sigma_{j(l)=1} m_{j(l)} = \Lambda;\ M_1 + \Lambda = M.$$

Further assume that there exists a $\mathbf{w}^*$ with positive entries such that

(i) $<\mathbf{w}^*, \mathbf{x}_j> \geq \alpha$ for j= 1,2, …, p
(ii) $<\mathbf{w}^*, \mathbf{e}_j> > 0$ for j= 1,2, …, n
(iii) $||\mathbf{w}^*|| = \alpha / \Delta_{opt}$

Then the following bounds for $<\mathbf{w}^*, \mathbf{w}_M>$ are obtained:

Lower bound: $<\mathbf{w}^*, \mathbf{w}_M> \geq M_1^* \alpha + \Lambda^* \min_{1 \leq k \leq n} \mathbf{w}^*[k] \geq M^* \alpha_1$ (1)

where $\alpha_1 := \min\{ \alpha, \min_{1 \leq k \leq n} \mathbf{w}^*[k]\}$
Upper bound: $||\mathbf{w}_M||^2 - ||\mathbf{w}_{M-1}||^2 = 2<\mathbf{w}_{m-1}, \mathbf{x}_{j(m-1)}> + || \mathbf{x}_{j(m-1)}||^2 \leq 2\alpha + B$ say, where $B:=\max\{\max_{1 \leq k \leq p} || \mathbf{x}_k||^2, 1\}$
Hence $||\mathbf{w}_M|| \leq \{M(2 \alpha + B)\}^{1/2}$.
And thus by the Cauchy-Schwartz inequality

$$<\mathbf{w}^*, \mathbf{w}_M> \leq ||\mathbf{w}^*|| \{M(2\alpha + B)\}^{1/2} \quad (2)$$

(1) and (2) together imply convergence since $M^{1/2} \leq ||\mathbf{w}^*|| (2\alpha + B)^{1/2} / \alpha_1$.

Optimal Margin of Separation

In order to show that the algorithm gives the optimal margin of separation as $\alpha \rightarrow \infty$ one needs an additional assumption (note that in view of the experimental results given above this does not seem too restrictive), namely additional assumption: $\alpha = \alpha_1$.
As in [14] decompose $\mathbf{w}_t$ as $\mathbf{w}_t = a(t)\mathbf{w}^* + \mathbf{k}_t$ where $<\mathbf{k}_t, \mathbf{w}^*> = 0$.
Argue as above but reason separately for $\mathbf{w}^*$ and $\mathbf{k}_t$ by decomposing $\mathbf{x}_{j(t)}$ accordingly.
First note that $a(t) = <\mathbf{w}_t, \mathbf{w}^*>/||\mathbf{w}^*||^2$ and hence that $a(t) > 0$ for $t \geq 1$ because of assumptions (i) and (ii) for $||\mathbf{w}^*||$.
Further note that $\mathbf{x}_{j(t)}$ always has a negative projection on $\mathbf{k}_t$, i.e. $< \mathbf{k}_t, \mathbf{x}_{j(t)} > < 0$, since this has been shown for $\mathbf{x}_{j(t)} = \mathbf{x}_k$ for some k satisfying $1 \leq k \leq p$ in [9],[14], and for $\mathbf{x}_{j(t)} = \mathbf{e}_l$ for some l satisfying $1 \leq l \leq n$ it follows since then $<\mathbf{w}_t, \mathbf{e}_l> = a(t)<\mathbf{w}^*, \mathbf{e}_l> + <\mathbf{k}_t, \mathbf{e}_l> \leq 0$, where $a(t)<\mathbf{w}^*, \mathbf{e}_l> > 0$ because of the positivity of a(t) and property (ii) of $||\mathbf{w}^*||$.
Moreover $\mathbf{w}_{t+1} = \mathbf{w}_t + \mathbf{x}_{j(t)} =$

$$[a(t) + < \mathbf{x}_{j(t)}, \mathbf{w}^*>/||\mathbf{w}^*||^2] \mathbf{w}^* + [1 + < \mathbf{x}_{j(t)}, \mathbf{k}_t>/|| \mathbf{k}_t ||^2] \mathbf{k}_t$$

whence $||\mathbf{k}_t||^2 - ||\mathbf{k}_{t-1}||^2 = 2< \mathbf{x}_{j(t-1)}, \mathbf{k}_{t-1}> + || \mathbf{x}_{j(t-1)} ||^2 \leq B$.

Thus $||\mathbf{k}_t||^2 \leq t^* B$ (3)

If learning stops after M steps then a(M-1) can be bounded as follows

$$<\mathbf{w}_{M-1}, \mathbf{x}_{j(M-1)}> = a(M-1) <\mathbf{w}^*, \mathbf{x}_{j(M-1)}> + <\mathbf{k}_{M-1}, \mathbf{x}_{j(M-1)}> < \alpha.$$

Hence, using the additional assumption and property (i) of $\mathbf{w}^*$ one obtains

$$a(M-1) < [\alpha + |<\mathbf{k}_{M-1}, \mathbf{x}_{j(M-1)}> |]/ \alpha.$$

From this it follows that

$$a(M-1) \leq [\alpha + ||\mathbf{k}_{M-1}||^* || \mathbf{x}_{j(M-1)}||]/ \alpha \leq [\alpha + (M-1)^{1/2} B]/ \alpha \quad (4)$$

A bound on a(M) is then obtained via

$$a(M) - a(M-1) = <\mathbf{w}_M - \mathbf{w}_{M-1}, \mathbf{w}^*>/||\mathbf{w}^*||^2 = |< \mathbf{x}_{j(M-1)}, \mathbf{w}^*>|/||\mathbf{w}^*||^2 \leq B^{1/2}/||\mathbf{w}^*||$$

$$= B^{1/2}\Delta_{opt}/\alpha \quad (5)$$

Hence, using (4) and (5), $a(M) \leq [\alpha + (M-1)^{1/2}B]/\alpha + B^{1/2}\Delta_{opt}/\alpha$ (6)

Using (6) gives

$$\|\mathbf{w}_M\| = a(M)\|\mathbf{w}^*\| + \|\mathbf{k}_M\| \leq \{[\alpha + (M-1)^{1/2}B]/\alpha + B^{1/2}\Delta_{opt}/\alpha\}\|\mathbf{w}^*\| + MB^{1/2}$$

$$= \|\mathbf{w}^*\|\{[\alpha + (M-1)^{1/2}B]/c + B^{1/2}\Delta_{opt}/\alpha + (\Delta_{opt}/\alpha)\, MB^{1/2}\} \quad 7)$$

Finally from (7) one obtains $\alpha/\Delta_{opt} \leq \alpha/\Delta_\alpha \leq \|\mathbf{w}_M\| \rightarrow \|\mathbf{w}^*\| = \alpha/\Delta_{opt}$ as $\alpha \rightarrow \infty$ since M grows at most linearly with α.