MINIMAL DISTORTION MAPPINGS OF SURFACES FOR MEDICAL IMAGES

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Abstract: In this paper we present a simple method for minimal distortion development of triangulated surfaces for colon mapping and general analysis of medical images. The method is based on classical results of Gehring and Väisalä regarding the existence of quasi-comformal and quasi-isometric mappings between Riemannian manifolds. Random and curvature based variations of the algorithm are presented. In addition the algorithm enables the user to compute the maximal distortion errors. The algorithm was tested both on synthetic images of the human skull and on real CT images of the human colon.

1 INTRODUCTION

Two-dimensional representation by flattening of three-dimensional object scans are of paramount importance, in many medical applications of image processing, such as medical imaging for noninvasive diagnosis and image guided surgery. For example, it is often advantageous to present three-dimensional MRI or CT scans of the cortex as flat two-dimensional images. Yet in order to do so in a meaningful manner, so that the diagnosis will be accurate, it is essential that the geometric distortion, in terms of change of angles and lengths, caused by this representation, will be minimal. However, since usually the studied surfaces is not isometric to the plane, a zero-distortion solution is seldom feasible. Yet, a reasonable solution to this problem is given by conformal maps (Haker et. al., 2000a), (Haker et. al., 2000b). Mapping the surface conformally to the (complex) plane preserves angles and therefore the local shape. Since this cannot be achieved in a global way, all solutions are local.

In spite of previous claims, flattening by means of conformal mapping is practically not possible, particularly so for highly folded surfaces, such as the colon and the cortex. Hence some distortion should be expected in cases encountered in medical imaging. Therefore, one should expect to obtain, some bounded amount of distortion. This can be achieved by quasi-isometric/quasi-conformal maps (i.e. maps that are almost isometries/conformal; precise definition will follow in Section 3). Practically, there is a tradeoff between the cost of an implementation on one hand and accuracy on the other. Common to all solutions is the fact, which cannot be avoided because of the inevitable distortion, that the more locally one is willing to focus, the more accurate the results become.

2 RELATED STUDIES

Previously-published approaches to the problem of minimal distortion flattening of surfaces can be sorted into three categories:

2.1 Variational Methods

Haker et al., (Haker et. al., 2000a), (Haker et. al., 2000b) S. introduced the application of a variational method in conformal flattening of CT and MRI 3-D scans of the brain and colon for the purpose of medical imaging. The method is essentially done by solving Dirichlet problem for the Laplace-Beltrami operator $\triangle u = 0$ on a given surface Σ, with certain (Dirichlet) boundary conditions on $\partial\Sigma$. A solution to this problem is a harmonic (thus conformal) map from the surface to the (complex) plane. The solution presented in (Haker et. al., 2000a) and (Haker et. al., 2000b) is a piecewise linear (PL) approximation of the smooth solution, achieved by solving a proper system of linear equations.

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2.2 Circle Packing

In (Hurdal et. al., 1999) Hurdal a method of building such a conformal map using circle packing is suggested. This relies on the ability to approximate conformal structure on surfaces by circle packings. The authors use this method for MRI brain images and conformally map them to the three possible models of geometry in dimension 2 (i.e. the 2-sphere, the Euclidian plane and the hyperbolic plane). Yet, the method is applicable for a surface which are topologically equivalent to a disk whereas the brain cortex surface is not. This means that there is a point of the brain (actually a neighborhood of a point), which will not map conformally to the plane, and in this neighborhood the dilatation will be infinitely large. Moreover, it must be assumed that the surface triangulation is homogeneous in the sense that all triangles are equilateral. Such triangulations are seldom attainable.

2.3 Holomorphic 1-forms

Gu et al. ((Gu, X. and Yau, S. T., 2002) and a series of consequent papers), are using holomorphic 1-forms in order to compute global conformal structure of a smooth surface of arbitrary genus and arbitrary number of boundary components. Holomorphic 1-forms are differential forms that depict conformal structure. As such, this method can be applied to colon unfolding. Yet, implementation of this method is extremely time consuming.

2.4 Angle Methods

Sheffer et al. (Sheffer, A. de Stuler, E.) parameterize surfaces via an angle-based method that minimizes, in a way, angle distortion while flattening. However, the surfaces are assumed to be approximated by cone surfaces i.e. one considers surfaces that are composed of cone-like neighborhoods.

2.5 The Proposed Method

We propose yet another solution to the problem of surface unfolding, with a special focus on its applications in medical imaging. The method proposed herein is based on theoretical results obtained by Gehring and Väisalä in the 1960‘s and refereed to in (Gehring-Väisala, 1965). Gehring and Väisalä investigated the existence of quasi-conformal maps between Riemann manifolds. The basic advantages of the proposed method is its simplicity. It is straightforward to implement this method and it is computationally efficient. An additional advantage is that it is possible to guarantee that the distortion will not exceed a predefined bound, which can be as small as desired with respect to the extent of localization one is ready to accept. Moreover, since, in contrast to other methods, the algorithm makes no use to derivatives, it is highly suitable for analysis of noisy data and for the study of surfaces with folds and ridges, such as the colon wall and the cortex. The proposed algorithm is most able in cases where the surface is complex (high and non-constant curvature) such as colon wrapping.

The paper is organized as follows: In the next section we provide a review of related works. In Section 3 some theoretical background to the fundamental work of Gehring and Väisalä. In the following section we describe our algorithm for surface flattening, based on their ideas. In Section 5 we present some experimental results of this scheme and in Section 6 we summarize the paper and discuss further improvements.

3 THEORETICAL BACKGROUND

Definition 1 Let $D \subset \mathbb{R}^3$ be a domain. A homeomorphism $f : D \to \mathbb{R}^3$ is called a *quasi-isometry* (or a *bi-Lipschitz mapping*), iff $\frac{1}{C}|p_1 - p_2| \leq |f(p_1) - f(p_2)| < C|p_1 - p_2|$, for all $p_1, p_2 \in D$; $1 \leq C < \infty$. $C(f) = \min\{C \mid f \text{ is a quasi} - \text{isometry}\}$ is called the minimal distortion of f (in D). (The distances considered are the induced intrinsic distances on the surfaces.)

If f is a quasi-isometry then $K_I(f) \leq C(f)^2$ and $K_O(f) \leq C(f)^2$, where $K_I(f)$ and $K_O(f)$ represent the *inner* and *outer dilatation* of f, respectively (see (Caraman, P., 1974) and references therein). It follows that any quasi-isometry is a quasi-conformal mapping (while, evidently, not every quasi-conformal mapping is a quasi-isometry). Quasi-conformal is the same as quasi-isometry where distances are replaced by inner products between tangent vectors.

Definition 2 Let $S \subset \mathbb{R}^3$ be a connected surface. S is called *admissible* iff for any $p \in S$, there exists a quasi-isometry i_p such that for any $\varepsilon > 0$ there exists a neighbourhood $U_p \subset \mathbb{R}^3$ of p, such that $i_p : U_p \to \mathbb{R}^3$ and $i_p(S \cap U_p) = D_p \subset \mathbb{R}^2$, where D_p is a domain and such that $C(i_p)$ satisfies: (i) $\sup_{p \in S} C(i_p) < \infty$ and (ii) $\sup_{p \in S} C(i_p) < 1 + \varepsilon$.

Let S be a surface, $\vec{n}$ be a fixed unitary vector, and $p \in S$, such that there exists a neighbourhood $V \subset S$, such that $V \simeq D^2$, where $D^2 = \{x \in \mathbb{R}^2 \,\big|\, ||x|| \leq 1\}$. Moreover, suppose that for any $q_1, q_2 \in S$, the acute angle $\measuredangle(q_1 q_2, \vec{n}) \geq \alpha$. We refer to the last condition as *the Geometric Condition* or *Gehring Condition*.

Then for any $x \in V$ there is a unique representation of the form: $x = q_x + u\vec{n}$, where q_x lies on the plane through p which is orthogonal to $\vec{n}$ and $u \in \mathbb{R}$. We

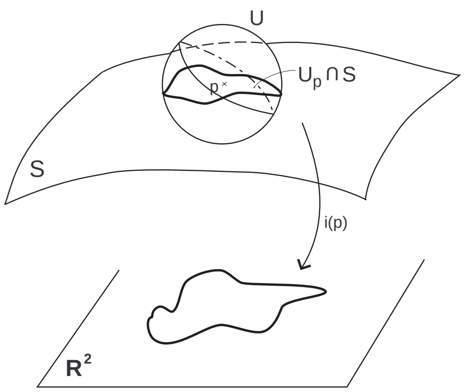

Figure 1: An Admissible Surface.

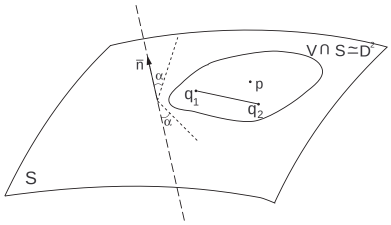

Figure 2: The Geometric Condition.

define: $Pr(x) = q_x$. (Note that $\vec{n}$ need not be the normal vector to S at p.)

We have that for any $p_1, p_2 \in S$ and any $a \in \mathbb{R}_+$ the following inequalities hold: $\frac{a}{A}|p_1 - p_2| \leq |Pr(p_1) - Pr(p_2)| \leq A|p_1 - p_2|$, where $A = \frac{1}{2}[(a \csc\alpha)^2 + 2a + 1]^2 + \frac{1}{2}[(a\csc\alpha)^2 - 2a + 1]^2$. In particular for $a = 1$ we get that $C(f) \leq \cot\alpha + 1$ and $K(f) \leq \left(\left(\frac{1}{2}(\cot\alpha)^2 + 4\right)^{\frac{1}{2}} + \frac{1}{2}\cot\alpha\right)^{\frac{3}{2}} \leq (\cot\alpha + 1)^{\frac{3}{2}}$; where $K(f) = \max\left(K_O(f), K_I(f)\right)$ is the *maximal dilatation* of f. We have thus obtained:

Theorem 1 (a) The quasi-isometric projection, Pr, yields a minimally distorted flat surface. (b) $C(f)$ and maximal dilatation $K(f)$: $C(f) \leq \cot\alpha + 1$.
(b) The maximal dilatation, $K(f)$, and distortion, $C(f)$, are bounded according to the following: $K(f) \leq (\cot\alpha + 1)^{\frac{3}{2}}$.

From the discussion above we conclude that $S \subset \mathbb{R}^3$ is an admissible surface if for any $p \in S$ there exists $\vec{n}_p$ such that, for any $q_1, q_2 \in U_p$ close enough to p, the acute angle $\measuredangle(q_1q_2, \vec{n}_p) \geq \alpha$. In particular, any smooth surface in $S \in \mathbb{R}^3$ is admissible.

3.1 The Algorithm

We proceed to present the algorithm that is used for obtaining a quasi-isometric (flat) representation of a given surface.

Assume that the surface is equipped with some triangulation T. Let N_p stand for the normal vector to the surface at a point p on the surface.

A triangle Δ, of the triangulation must be chosen. We project a patch of the surface quasi-isometrically onto the plane included in Δ. This patch is called the patch of Δ, and it consists of at least one triangle, Δ itself. There are two possibilities to chose Δ, one is in a random manner and the other is based on curvature considerations. We will refer to both ways later. For the moment, assume Δ was somehow chosen. After Δ is (trivially) projected onto itself we move to its neighbors. Suppose Δ' is a neighbor of Δ having edges e_1, e_2, e_3, where e_1 is the edge common to both Δ and Δ'.

We consider Δ' to be *Gehring compatible w.r.t* Δ, if the maximal angle between e_2 or e_3 and N_Δ (the normal vector to Δ), is greater then a predefined measure suited to the desired predefined maximal allowed distortion, i.e. $\max\{\measuredangle(e_2, N_\Delta), \measuredangle(e_3, N_\Delta)\} \geq \alpha$. We project Δ' *orthogonally* onto the plane included in Δ and insert it to the patch of Δ, iff it is Gehring compatible w.r.t Δ.

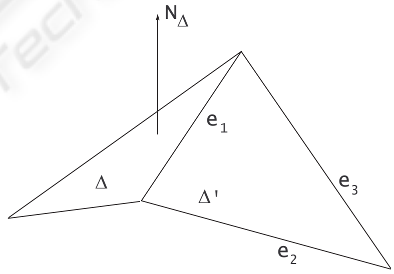

Figure 3: Gehring Compatible Triangles. Here a general projection direction N_Δ is depicted.

We keep adding triangles to the patch of Δ moving from an added triangle to its neighbors (of course) while avoiding repetitions, till no triangles can be added.

If by this time all triangles where added to the patch we have concluded. Otherwise, chose a new triangle that has not been projected yet, to be the starting triangle of a new patch. A pseudocode for this procedure can be easily written.

We conclude this section with the following remarks: One should keep in mind that the above given algorithm, as for any other flattening method, is local. Indeed, in a sense the (proposed) algorithm gives a measure of “globality” of this intrinsically local process. Our algorithm is best suited for highly folded surfaces, because of its intrinsic locality on the one hand and computational simplicity, on the other.

4 EXPERIMENTAL RESULT

We present some experimental results obtained by applying the algorithm presented in the previous section. In each of the examples both the input surface and a flattened representation of some patch are shown. Distortion error is computable according to the bounds given by *Theorem 1*. Details on the resolution of the mesh and also the number of patches may also be given.

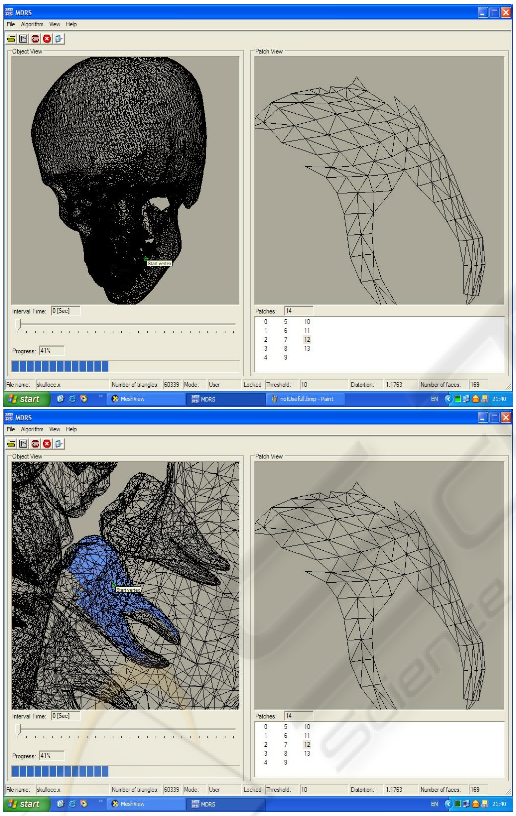

Figure 4: Tooth model: (a) and (b) have resolution of 60,339 triangles yet the resolution of the visualization is different. No change was caused to the flattening of the teeth. In both cases the dilatation is 1.1763.

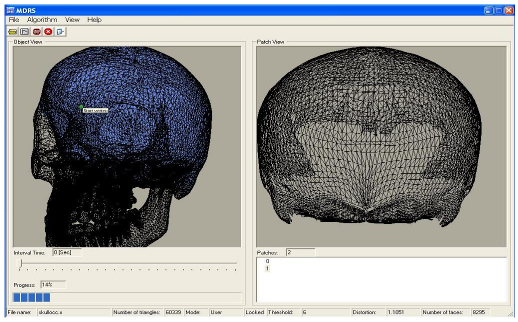

Figure 5: Skull model: resolution of 60,339 triangles. Initial triangle in the frontal region. Here $\alpha = 6^0$ and the dilatation is 1.1051.

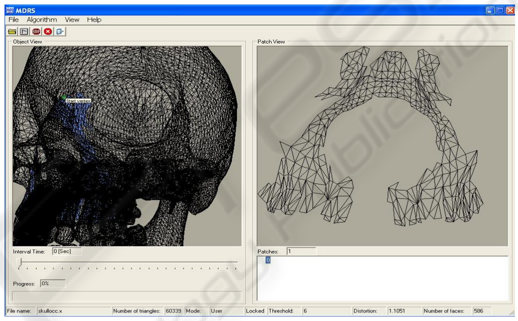

Figure 6: Skull model: resolution of 60,339 triangles. Initial triangle in the sinuses region. Here $\alpha = 6^0$ and the dilatation is 1.1051.

5 CONCLUDING REMARKS

Our new quasi-conformal-based algorithm for flattened representation of polyhedral meshes has been successfully applied to colon imaging, with minimal dilatation introduced by the flattening. The algorithm is based upon the works of Gehring-V$\ddot{a}$isala and others concerning the existence of quasi-isometric/quasi-conformal mappings between Riemannian manifolds.

From the implementation results it is evident that this algorithm, while being simple to program as well as efficient, yields good flattening results and maintains small dilatations even in areas where curvature is large and good flattening is a challenging task. Moreover, since there is a simple way to assess the resulting dilatation, the algorithm is implementable in such a way that the user can set in advance an upper bound on the resulting dilatation.

An additional advantage of the presented algorithm resides in the fact that in contrast to some of the related works, no use of derivatives is made. Consequently the algorithm is applicable to noisy data as well.

The main issue which calls for further study is related to the inherent locality of the quasi-conformal and quasi-isometric geometric mapping adopted in our approach. It is important to incorporate properties in a well-defined and precise manner. Such a generalized approach, which is currently under investigation,

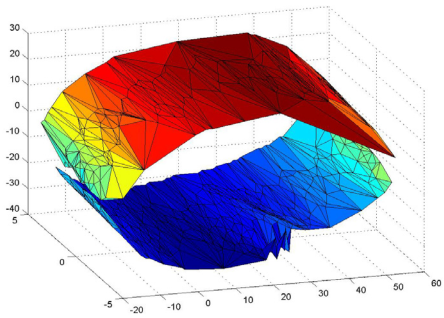

Figure 7: Colon CT-Images: Triangulated colon surface taken from 3 slices of human colon scan. Images are in curtesy of Dr. Doron Fisher from Rambam Madical Center in Haifa.

will permit “gluing” two neighbouring patches while keeping fixed bounded dilatation.

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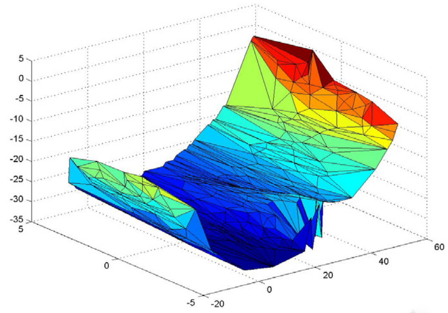
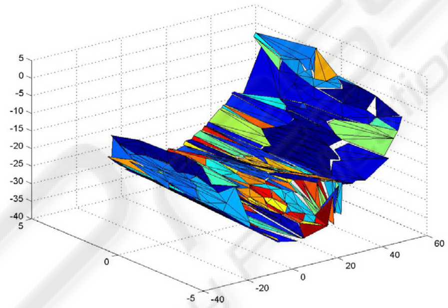


(a)



(b)

Figure 8: Colon Flattening: One half of the colon taken from 3 slices. (a) before flattening and (b) after.

REFERENCES

Caraman, P. (1974) *n-Dimensional Quasiconformal (QCf) Mappings*. Editura Academiei Române, Bucharest, Abacus Press, Tunbridge Wells Haessner Publishing, Inc., Newfoundland, New Jersey.

Gehring, W. F. and Väisala, J. (1965). The coefficients of quasiconformality. In *Acta Math.* **114**, *pp. 1-70*.

Gu, X. and Yau, S. T. (2002). Computing Conformal Structure of Surfaces. In *Communications In Information and Systems, Vol. 2, No. 2, pp. 121-146*.

Gu, X. and Yau, S. T. (2003). Global Conformal Surface Parameterization In *Eurographics Symposium on Geometry Processing*.

Haker, S. Angenet, S. Tannenbaum, A. Kikinis, R. (2000) Non Distorting Flattening Maps and the 3-D visualization of Colon CT Images. *IEEE Transauctions on Medical Imaging, Vol. 19, NO. 7*.

Haker, S. Angenet, S. Tannenbaum, A. Kikinis, R. Sapiro, G. Halle, M. (2000) Conformal Surface Parametrization for Texture Mapping. *IEEE Transauctions on Visualization and Computer Graphics, Vol. 6, NO. 2*.

Hara, A., Johnson, C., Reed, J., Ehman, R., Ilstrup, D. (1996) Colorectal polyp detection with CT colography: two- versus three-dimensional techniques. *Radiology*, vol. 200, pp. 49-54.

Hurdal, M. K., Bowers, P. L., Stephenson, K., Sumners, D. W. L., Rehm, K., Schaper. K., Rottenberg, D. A. (1999) Quasi Conformally Flat Mapping the Human Crebellum, In *Medical Image Computing and Computer-Assisted Intervention -MICCAI'99, (C. Taylor and A. Colchester. eds), vol. 1679, pp. 279-289.* Springer-Verlag, Berlin.

Paik, D., Beaulieu, C., Jeffrey, R., Karadi C., and Napel, S. (1999) Visualizations modes for CT colonography using cylindrical and planar projections. Technical Report, Department of Radiology, Stanford University School of Medicine, Stanford, CA.

Peyé, G. and Cohan, L. (2005) Geodesic Computations for Fast and Accurate Surface Flattening. preprint.

Sheffer, A. de Stuler, E. (2001) Parametrization of Faceted Surfaces for Meshing Using Angle Based Flattening. In *Engineering with Computers, vol. 17, pp. 326-337*.

Sorkine, O., Cohen-Or, Daniel, Goldenthal, R. and Lischinski, D. (2005) Bounded-distorsion Piecewise Mesh Parametrization. preprint.

Väisälä, J. (1971). *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Mathematics 229, Springer-Verlag, Berlin - Heidelberg - New-York.

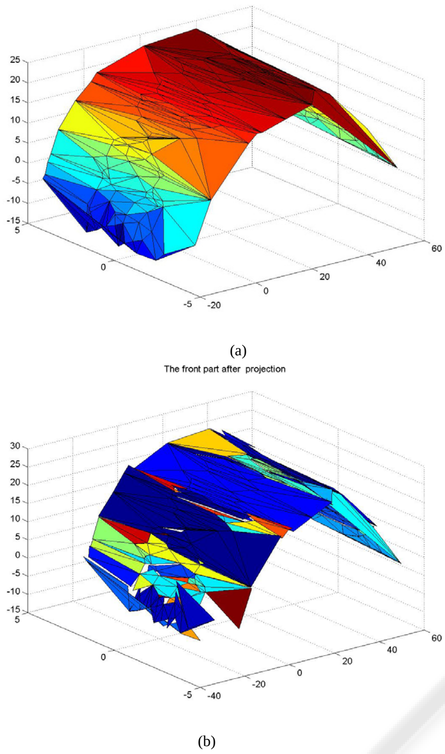


Figure 9: Colon Flattening: Second half of the colon slices. (a) before flattening and (b) flattened.