DETERMINING ELLIPSOIDAL BASINS OF ATTRACTION OF FUZZY SYSTEMS

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Abstract: This paper discusses how to obtain local stability results from a fuzzy system for which global ones cannot be obtained, basically due to infeasibility of some associated LMI problems. Two different approaches are compared: modifying the consequent models *vs.* setting up some relaxed LMI conditions if bounds on the memberships are known. Some examples are used to illustrate the approaches.

1 INTRODUCTION

In many literature contributions, LMI stability conditions (Boyd et al., 1994) are devised in order to prove stability and performance of Takagi-Sugeno (Takagi and Sugeno, 1985) fuzzy systems; however, such laws are usually independent of the values of membership functions, and fulfill for any arbitrary shapes of them (Tanaka and Wang, 2001; Wang et al., 1996). Knowledge of the shape of the membership functions may allow to lift some conservativeness.

For instance, if the usual Jacobian linearisation in $x = 0$ is stable, Lyapunov 1st theorem states that there exists a region in which the system is locally stable. The approach in this paper allows to explicitly define a minimum spherical zone around the equilibrium point where Lyapunov stability conditions are fulfilled, even in the case global quadratic-stability related LMIs are infeasible. Indeed, (Tanaka and Wang, 2001) shows that the basin of attraction for fuzzy systems may be membership dependent.

The structure of the paper is as follows: Next section discusses notation and widely-known stability theorems. Section 3 discusses a transformation of a fuzzy model when the membership functions are themselves a convex combination of some vertices. Section 4 applies the results to find the largest local quadratically stable region. Some examples are provided in Section 5, and a conclusion section summarises the main results.

2 PRELIMINARIES

Let us consider a Takagi-Sugeno (Takagi and Sugeno, 1985) (TS) fuzzy model:

$$\dot{x} = \sum_{i=1}^{n} \mu_i(x)(A_i \cdot x) \tag{1}$$

where μ_i represents membership functions such that:

$$\sum_{i=1}^{n} \mu_i(x) = 1, \quad \mu_i(x) > 0 \,\forall x \; i : 1 \ldots n$$

Stability of fuzzy systems

Lyapunov stability theory proves that such a system is stable if exist a function $V(x)$ such that:

$$V(x) > 0, \; \frac{dV}{dx} < 0, \; V(0) = 0, \; \forall \, x \neq 0$$

The analysis of the Lyapunov stability of TS fuzzy systems may be approached as a linear matrix inequality (LMI) optimization problem (Boyd et al., 1994). The most popular Lyapunov Functions proposed in literature are quadratic forms: $V(x) = x^T P x$. This type of Lyapunov functions fulfill the srability conditions if P is definite positive and if

$$\dot{V} = \sum_{i=1}^{n} \mu_i x^T (A_i^T P + P A_i) x < 0 \tag{2}$$

That holds if

$$A_i^T P + P A_i < 0, \; i : 1..n \tag{3}$$

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The above equation is an LMI, hence widely available LMI optimization software either finds a P or determines that the LMI is infeasible. The reader is referred to (Tanaka and Wang, 2001) for ample discussion.

Remark: Note that the membership functions μ do not appear in the LMI conditions. Hence, the same P defines a quadratic Lyapunov function for multiple nonlinear systems with the same "vertex models" as the original one. Such generality is a too restrictive condition that in some cases results in infeasibility being the underlying system actually stable.

When the above LMI problems are unfeasible, other alternative conditions must be sought. Fuzzy or piecewise Lyapunov functions are discussed in (Johansson, 1999), Fuzzy Lyapunov functions are discussed in (Oliveira et al., 1999).

A different alternative, in the authors' opinion, is trying to achieve *local* stability results in a zone around the equilibrium as large as possible. Such a result is motivated on the first Lyapunov theorem for local stability: if the linearised system in $x = 0$ is exponentially stable, then so it is the nonlinear one, for initial conditions in a sufficiently small neighborhood of $x = 0$.

3 LOCAL FUZZY MODELS

In order to analyze the local stability of a TS fuzzy model (1) within a region, the original model is modified using the information of the membership functions.

Lemma 1 *if the membership functions $\mu(x)$ of a fuzzy system described in* (1) *in a region of Ω can be themselves expressed as a convex sum of some vectors v_p:*

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p, \quad \forall\, x \in \Omega \tag{4}$$

where:

$$\mu(x) = [\mu_1(x), \mu_2(x), \ldots, \mu_n(x)]$$

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \; \beta_p(x) > 0 \; \forall x \in \Omega \; p : 1 \ldots n_v$$

Then the system can be transformed to:

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x \tag{5}$$

where

$$A_p^* = \sum_{i}^{n} v_{pi} A_i \tag{6}$$

Proof: *The expression* (4) *can be substituted in the system equation* (1)*:*

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p \tag{7}$$

$$v_p = [v_{p1}, v_{p2}, \ldots, v_{pn}] \tag{8}$$

$$\mu_i(x) = \sum_{p=1}^{n_v} \beta_p(x) v_{pi} \tag{9}$$

$$\dot{x} = \sum_{i=1}^{n} \sum_{p=1}^{n_v} \beta_p(x) v_{pi} A_i \cdot x \tag{10}$$

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) \sum_{i}^{n} v_{pi} A_i \cdot x \tag{11}$$

so the local representation of the system in Ω

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x \quad \forall x \in \Omega$$

where:

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \; \beta_p(x) > 0 \; \forall x \in \Omega \; p : 1 \ldots n_v$$

□

The convex-combination conditions for the membership functions required in the above lemmas are easy to meet. Indeed μ_i are assumed known in fuzzy systems. Then, the result below may be applied to obtain a (possibly conservative) vertex set.

Note 1 *Let us consider a region Ω. If bounds μ_i^M and μ_i^m on the extremum values of the membership functions in Ω can be computed, in such a way that:*

$$\mu_i^M \geq \max_{x \in \Omega} \mu_i(x) \quad \mu_i^m \leq \min_{x \in \Omega} \mu_i(x) \tag{12}$$

then there exist a set of $\beta_p(x)$, $p = 1, \ldots, n_v$ so that the vector of membership functions

$$\mu(x) = [\mu_1(x), \mu_2(x), \ldots, \mu_n(x)]$$

may be expressed in Ω as:

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p, \;\; x \in \Omega \tag{13}$$

where:

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \; \beta_p(x) > 0 \; \forall x \in \Omega \; p : 1 \ldots n_v$$

Indeed, the linear restrictions $\mu_i^M \geq \mu_i \geq \mu_i^m$, $\sum_i \mu_i = 1$ describe a bounded polytope with a finite number of vertices (Luenberger, 2003).

Well-known linear-programming-related methods to obtain the membership vector vertices may be used (related to the obtention of the basic feasible solutions in an LP problem (Luenberger, 2003)). A related alternative is described below.

Lemma 2 *Consider the set Σ_i of at most 2^{n-1} vectors defined by:*

$$\Sigma_i = \{[\tilde{\mu}_1, ..., \tilde{\mu}_{i-1}, X, \tilde{\mu}_{i+1}, ..., \tilde{\mu}_n], \quad X = 1 - \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \tilde{\mu}_j$$

such that $\tilde{\mu}_j \in \{\mu_j^M, \mu_j^m\}$ $j \neq i$, $\mu_i^m \leq X \leq \mu_i^M\}$ (14)

Then, the vectors belonging to the set

$$\Sigma = \bigcup_{i=1}^{n} \Sigma_i \tag{15}$$

satisfy (13) *for some* β_p.

Indeed, as there is only one equality restriction in memberships, all except one of them are "free" to attain an extremum value; the remaining one must fulfill the add-1 restriction *and* be inside its required bounds. The above lema produces the union of all the "all minus one" combinations, and the sought vertices will belong to such set.

Example. For instance, if three memberships have minimum and maximum values given by {0.15,0.3,0.35} and {0.6,0.5,0.4}, the set Σ_1 is originated by the four combinations:

$$\{(X_1, 0.3, 0.35), (X_2, 0.5, 0.35), (X_3, 0.3, 0.4), (X_4, 0.5, 0.4)\}$$

with $X_1 = 1 - 0.65 = 0.35$, $X_2 = 0.15$, $X_3 = 0.3$, $X_4 = 0.1$. As X_4 is out of the required range, the candidate vertices kept are:

$$\Sigma_1 = \{(0.35, 0.3, 0.35), (0.15, 0.5, 0.35), (0.3, 0.3, 0.4)\}$$

The set Σ_2 is generated by:

$$\{(0.15, X_1, 0.35), (0.6, X_2, 0.35), (0.15, X_3, 0.4), (0.6, X_4, 0.4)\}$$

with $X_1 = 0.5$, $X_2 = 0.05$, $X_3 = 0.45$ and $X_4 = 0$. Hence,

$$\Sigma_2 = \{(0.15, 0.5, 0.35), (0.15, 0.45, 0.4)\}$$

Regarding the third membership,

$$\{(0.15, 0.3, X_1), (0.6, 0.3, X_2), (0.15, 0.5, X_3), (0.6, 0.5, X_4)\}$$

results in

$$\Sigma_3 = \{(0.15, 0.5, 0.35)\}$$

hence the resulting set of vertices to compute the local models is:

$$\Sigma = \{(0.35, 0.3, 0.35), (0.15, 0.5, 0.35), (0.3, 0.3, 0.4), (0.15, 0.45, 0.4)\}$$

4 STABILITY ANALYSIS IN A ZONE

The knowledge of the membership functions will allow to obtain some local stability analysis results for a fuzzy systems. Two alternatives may be applied: the first one will use the above defined local models; the second one will use some relaxations on LMI conditions via additional variables and knowledge of the minimum and maximum bounds on membership.

4.1 Local Fuzzy Models

By using the transformed models discussed in the previous section, local stability results may be obtained by the lemmas in Section 2.

Lemma 3 *The ellipsoidal region $\Omega^* \subset \Omega$*

$$\Omega^* = \{x \backslash x^T P x \leq V_M, \ P > 0\} \tag{16}$$

is a basin of attraction of the equilibrium point $x = 0$ of the system (1) *if*

$$\tag{17}$$

$$V_M \quad \leq \quad min\{x^T P x \backslash x \in \ \partial\Omega\} \tag{18}$$

where $\partial\Omega$ denotes the boundary of Ω and P verifies:

$$A_p^{*T} P + P A_p^* < 0 \ \ p : 1, \ldots, n_v \tag{19}$$

i.e., all trajectories with initial state in Ω^* converge assymptotically to $x = 0$.

Proof: *As, by Lemma 1, the system can be expressed in Ω as:*

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x$$

if the LMI (19) *is feasible for a positive definite matrix P, $V(x) = x^T P x$ is a decreasing function with time, so a Lyapunov function has been obtained ensuring that Ω^* is an invariant set. La Salle's theorem (Khalil, 1996) ensures that every solution starting in Ω^* will approach $x = 0$.*

As the expression of the local system (5) *is not valid outside Ω, then the local stability can only be proved in the largest ellipsoid Ω^* contained in Ω, which will be defined by a value of V_m equal to the minimum value of $V(x)$ in the boundary of Ω ($\partial\Omega$).* □

The following lemma is useful in order to set up an LMI characterisation of the largest ellipsoid in Ω which is a Lyapunov equipotential[1].

Suppose Ω defined as a symmetric polytope that contains $x = 0$:

$$\Omega = \{x\backslash |a_i^T x| \leq 1 i : 1, ..., n_p\} \tag{20}$$

Lemma 4 $\Theta = \{x\backslash x^T Q^{-1} x \leq 1\}$, $Q = Q^T > 0$ *is an ellipsoid contained in* Ω *which itself contains the maximum volume sphere centered at* $x = 0$ *if the LMI problem*

minimize λ
subject to $\lambda I > Q^{-1} > 0$
$Q > 0, a_1^T Q a_i \leq 1, \;\; i = 1, ...n_p$

is feasible. Then, no other ellipsoid in Ω *contains a larger centered sphere.* □

The proof appears in (Boyd et al., 1994) chapter 3.

Theorem 1 *Consider the system* (1). *The largest spherical basin of attraction of* $x = 0$ *provable by a quadratic Lyapunov function in a symmetric polytopic region* Ω *has a radius* $\lambda^{-\frac{1}{2}}$ *given by the solution of the following LMI problem:*

minimize λ *subject to*

$$\begin{aligned} \lambda I > P \;\; &> \;\; 0 && (21)\\ P \;\; &> \;\; 0 && (22)\\ \begin{pmatrix} P & a_j \\ a_j^T & 1 \end{pmatrix} \;\; &> \;\; 0, \; j : 1...n_p && (23)\\ A_p^{*T} P + P A_P^* \;\; &< \;\; 0, \; p : 1...n_v && (24) \end{aligned}$$

and Ω *is defined as* (20). *The ellipsoid* $\Theta = \{x\backslash x^T P x \leq 1\}$ *is, of course, also contained in the basin of attraction of* $x = 0$.

Proof: *Conditions 24 imply that trajectories inside any equipotential region defined by* P *converge to the point* $x = 0$*, as shown in Lemma 3.*

Applying the Schur complement, the conditions (23) *are equivalent to*

$$a_p^T P^{-1} a_p < 1, i : 1..n_p$$

Then, conditions (23) *keep* Θ *inside* Ω *and the condition* (21) *along with the LMI objective, maximize the radius of the quadratically invariant sphere contained in* Θ*, from Lemma 4.* □

[1]Largest is here understood as containing the largest spherical ball around $x = 0$, i..e, guaranteeing stability for the largest initial distance to the origin.

4.2 Relaxed LMI Conditions

Another way to approach the problem is relaxing the LMI conditions using que information about the membership functions μ_i in the zone Ω in which local stability is studied. This will allow to express some results (possibly more conservative than the previous one) using the minimum and maximum values of memberships in the zone (or some bounds on them), without the need of calculating transformed local models.

Assume that, in the zone Ω, the limits of μ_j are

$$\mu_j^m \leq \mu_j \leq \mu_j^M \tag{25}$$

Then, for any positive $\tau \in \mathcal{R}$:

$$\mu_j \tau \leq \mu_j^M \tau = \mu_j^M \sum_{i=1}^{n} \mu_i \tau \tag{26}$$

where $\sum_{i=1}^{n} \mu_i = 1$ has been used in the equality. Then, for any positive definite N_j^M:

$$\mu_j x^T N_j^M x \leq \mu_j^M \sum_{i=1}^{n} \mu_i x^T N_j^M x \tag{27}$$

Hence, the term

$$\sum_{i=1}^{n} \mu_i \mu_j^M x^T N_j^M x - \mu_j x^T N_j^M x > 0$$

may be added to the stability condition (3), so that if

$$\sum_{i=1}^{n} \mu_i x^T (A_i^T P + P A_i) x + \sum_{i=1}^{n} \mu_i \mu_j^M x^T N_j^M x \\ - \mu_j x^T N_j^M x < 0 \tag{28}$$

then, the equation (2) holds. Reordering the terms, the LMI conditions below are obtained:

$$(A_i^T P + P A_i) + \mu_j^M N_j^M < 0, \; \forall i \neq j \tag{29}$$

$$(A_j^T P + P A_j) - (1 - \mu_j^M) N_j^M < 0, \; N_j^M > 0 \tag{30}$$

and adding the condition $\mu_j^m \leq \mu_j$, for any positive symmetric matrix N_j^m, the expression

$$\mu_j x^T N_j^m x - \sum_{i=1}^{n} \mu_i \mu_j^m x^T N_j^m x > 0 \tag{31}$$

can be proved analogously to the maximum N_j^M case. Then (2) is positive if

$$\begin{aligned} (A_i^T P + P A_i) + \mu_j^M N_j^M \\ - \mu_j^m N_j^m < 0 \quad \forall i \neq j \end{aligned} \tag{32}$$

$$\begin{aligned} (A_j^T P + P A_j) - (1 - \mu_j^M) N_j^M \\ + (1 - \mu_j^m) N_j^m < 0, \quad N_j^M, N_j^m > 0 \end{aligned} \tag{33}$$

Note that, in the above expressions, j is a fixed number. If a bound of μ_j is known for all j, the theorem below can be proved.

Theorem 2 *Consider the system* (1). *The largest spherical basin of attraction of* $x = 0$ *provable by a quadratic Lyapunov function in a symmetric polytopic region* Ω *has a radius* $\lambda^{-\frac{1}{2}}$ *given by the solution of the following LMI problem in the variables* P, N_i^m, N_i^M*:*

minimize λ *subject to*

$$\lambda I > P > 0 \quad (34)$$

$$P > 0 \quad (35)$$

$$\begin{pmatrix} P & a_j \\ a_j^T & 1 \end{pmatrix} > 0,\ j : 1...n_p \quad (36)$$

$$A_i{}^T P + P A_i - (1-\mu_i^M) N_i^M + (1-\mu_i^m) N_i^m + \sum_{j \neq i} (\mu_j^M N_j^M - \mu_j^m N_j^m) < 0,\ i : 1...n \quad (37)$$

and Ω *is defined as* (20). *The ellipsoid* $\Theta = \{x \backslash x^T P x \leq 1\}$ *is, of course, also contained in the basin of attraction of* $x = 0$. □

4.3 Algorithm

The results in previous sections may be combined in order to obtain an algorithm to compute the largest ball around $x = 0$ for which attraction is ensured.

Basically, the procedure will first check the extreme cases: (1) checking for feasibility of LMI problems as stated in Section 2 (2) checking for stability of the linearised model around $x = 0$.

If the first one is unfeasible but the second one is feasible, selecting a polytopic region on the state space and a scaling factor ρ allows to set up a bisection procedure in order to determine the largest feasible ρ.

5 EXAMPLES

Example 1. Let us have a fuzzy system given by:

$$\dot{x} = \sum_{i=1}^{2} \mu_i(x) A_i x \quad (38)$$

$$A_1 = \begin{bmatrix} -0.5 & -1 \\ -1 & -0.5 \end{bmatrix} \quad (39)$$

$$A_2 = \begin{bmatrix} -0.5 & 1 \\ 1 & -0.5 \end{bmatrix} \quad (40)$$

Figure 1 shows the membership functions μ_1 and μ_2 which, for simplicity, depend only on x_2. The value of $a = 1$ will be assumed.

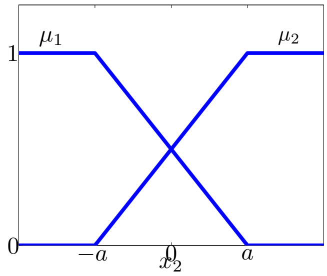

Figure 1: Membership functions $\mu_1(x_2)$, $\mu_2(x_2)$.

Define Ω_k as a rectangle bounded in x_2, unbounded in x_1:

$$\Omega_k = \{x \backslash\ |(0\ 1/\rho_k)x| \leq 1\}$$

where k is the iteration number.
Note that the maximum and minimum values of μ_i in Ω are easily obtained, and the Lemma 4 can be applied.

In the proposed procedure, the LMIs for $\rho = 1$ are unfeasible. However, the linearised model is:

$$\dot{x} = (0.5A_1 + 0.5A_2)x = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \quad (41)$$

which is stable. Hence, there exists a zone around $x = 0$ (possibly small) where local stability holds. The procedures in this paper allow to determine the largest sphere around $x = 0$ for which local quadratic stability holds.

Let us consider for the fist iteration $\rho_1 = 0.1$. The maximum and minimum values of μ are, in that case: $\mu_1^M = 0.55$, $\mu_1^m = 0.45$, $\mu_2^M = 0.55$, $\mu_2^m = 0.45$
Then the vertices obtained in the region Ω_1 are:

$$v_1 = [\ 0.45 \quad 0.55\ ]$$
$$v_2 = [\ 0.55 \quad 0.45\ ]$$

The local fuzzy model from Lemma 1 is described by:

$$A_1^* = \begin{bmatrix} -0.5 & 0.1 \\ 0.1 & -0.5 \end{bmatrix}$$
$$A_2^* = \begin{bmatrix} -0.5 & -0.1 \\ -0.1 & -0.5 \end{bmatrix}$$

And, solving the LMIs:

$$A_1^{*T} P + P A_1^* < 0$$
$$A_2^{*T} P + P A_2^* < 0$$
$$X > 0$$

local stability in a certain ellipsoidal region inside Ω_1 is proved.

When the same procedure is applied to $\rho = 0.5$ the LMIs are unfeasible. The LMIs are, however, feasible

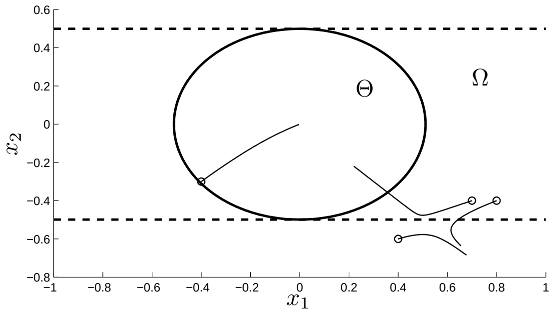

Figure 2: The Basin of attraction of $x = 0$ provable by quadratic stability (Example 1).

for any $\rho < 0.5$. for instance, $\rho_n = 0.499$ results in the following LMI conditions:

$$\begin{aligned} A_1^{*T}P + PA_1^* &< 0 \\ A_2^{*T}P + PA_2^* &< 0 \\ \begin{pmatrix} P & & \begin{matrix} 0 \\ 1/\rho_n \end{matrix} \\ 0 & 1/\rho_n & 1 \end{pmatrix} &> 0 \\ P &> 0 \\ \lambda I &> P \end{aligned}$$

which are feasible for the above value of ρ_n and, the matrix P obtained for the minimum λ defines an ellipsoidal basin of atraction $\Theta = \{x \backslash x^T P x \leq 1\}$ with

$$P = \begin{bmatrix} 3.8274 & 0 \\ 0 & 4.016 \end{bmatrix}$$

Figure 2 depicts the obtained basin of attraction, and some examples of trajectories from a set of starting points. The maximum circle found by the LMI toolbox has a radius of 0.499.

Trajectories with starting points inside the ellipsoid Θ are guaranteed to converge to the origin. Note that the ellipsoid obtained by the LMIs is not a circle. Points outside the ellipsoid may lead to either convergent or non-convergent trajectories.

Example 2. Let us considerer the same fuzzy model as in Example 1. Then applying the theorem 2,

minimize λ subject to

$$\begin{aligned} &(A_1^T P + PA_1) - (1-\mu_1^M)N_j^M + (1-\mu_1^m)N_1^m \\ &\qquad + \mu_2^M N_2^M - \mu_2^m N_2^m < 0 \quad \forall i \neq j \\ &(A_2^T P + PA_2) - (1-\mu_2^M)N_j^M + (1-\mu_2^m)N_2^m \\ &\qquad + \mu_1^M N_1^M - \mu_1^m N_1^m < 0 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P & & \begin{matrix} 0 \\ 1/\rho_n \end{matrix} \\ 0 & 1/\rho_n & 1 \end{pmatrix} &> 0 \\ P, N_1^M, N_1^m, N_2^M, N_2^m &> 0 \\ \lambda I &> P \end{aligned}$$

Conveniently, we take the same region shape Ω that in Example 1, the limits μ_i^m and μ_j^M are the minimum and maximum value of μ_i in the region Ω. The maximum ρ obtained is 0.26, *i.e.* 0.24 units less than the obtained in the previous example. From this example, the conditions discussed in Section 4.2 seem more conservative than those in Section 4.1.

6 CONCLUSIONS

This paper shows how *local* stability results (the largest sphere around $x = 0$ for which a quadratic Lyapunov function can be proven via LMI) may be obtained in fuzzy systems via the knowledge of the membership functions, even when no feasible quadratic Lyapunov function can be found to prove *global* stability. The found sphere is part of a larger ellipsoidal guaranteed basin of attraction.

In this way, if the linearised system around the equilibrium is stable, a precise characterisation of the local stability region stated in Lyapunov 1st theorem is achieved.

The approach based on relaxed LMI conditions from membership bounds yields more conservative results but it is simpler, without the need of changing the Takagi-Sugeno consequents.

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