Fast Algorithm for Optimal Polygonal Approximation of Shape Boundaries

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Abstract. This paper presents fast optimal algorithm for approximation of a shape boundary with a polygon having minimum number of vertices for a given maximum tolerable approximation error. For this purpose, the directed acyclic graph (DAG) formulation of the polygonal approximation problem is considered. The reduction in computational complexity is achieved by reducing the number of admissible edges in the DAG and speeding up the process of determining whether the edge distortion is within the tolerable limit. The proposed algorithm is compared with other optimal algorithms in terms of the execution time.

1 Introduction

Representation of shape boundaries is of great interest in a number of fields such as object-based video coding, video content retrieval based on object descriptions, object recognition etc. The efficient way to represent shape boundaries is the polygonal approximation. The optimality of polygonal representation with respect to number of vertices is relevant in applications involving pattern analysis, recognition, matching and search and retrieval because in these applications, the speed of algorithms is proportional to the number of vertices of the polygon.

The classical method for polygonal approximation is the iterative refinement method (IRM) [1][2] in which a shape boundary is recursively split into polygon edges until the maximum deviation between the boundary and the polygon lies below a predefined error threshold. However, IRM is not the optimal solution because it does not always yield the minimal number of polygon vertices. Several methods have been proposed for polygonal approximation that provide strictly optimal solutions according to a certain optimization criterion. A scan-along algorithm for optimum polygon approximation of planar curves that yields the minimal number of edges is presented in [3]. A dynamic programming algorithm for optimal polygon approximation is presented in [4]. Recently in [5], the rate-distortion optimized polygonal approximation is obtained by formulating the problem as finding the shortest path in a single source weighted directed acyclic graph (DAG). The optimal approaches are in general computationally intensive and are not suitable for real-time applications. Therefore, reducing the computational complexity of optimal approaches is very important.

I. Hosur P. and A. Carrasco R. (2005).
Fast Algorithm for Optimal Polygonal Approximation of Shape Boundaries.
In *Proceedings of the 5th International Workshop on Pattern Recognition in Information Systems*, pages 104-113
DOI: 10.5220/0002579501040113

In the DAG formulation of the optimal polygonal approximation problem, the computational complexity can be decreased by reducing the number of edges in the DAG. In the sliding window method proposed in [6], the number of edges in the DAG formulation are reduced by considering only those edges from each vertex which lie within a window of predefined size starting from that vertex. However, the ad-hoc window size may yield sub-optimal results as demonstrated through our experimental results presented in Section 6. Another method for reducing the edges in the DAG formulation is proposed in [7]. This method utilizes the fact that, as we scan-along a shape boundary, there can no longer be any admissible edge beyond the boundary scan-point at which the edge distortion exceeds twice the value of the error threshold.

This paper presents an algorithm for optimal polygonal approximation of shape boundaries that yields significantly better speed-up performance as compared to other optimal algorithms. The main idea of the proposed algorithm is to reduce the complexity associated with the computation of edge distortion in addition to the reduction of the number of edges.

The paper is organized as follows. Section 2 states the problem of optimal polygonal approximation. The DAG formulation of the problem is introduced in Section 3. In Section 4, the reference algorithms for optimal polygonal approximation are explained. The proposed algorithm is described in Section 5. The performance of the proposed algorithm is compared with that of the reference algorithms in Section 6. The conclusions are given in Section 7.

2 Problem Statement

Suppose a shape boundary is represented by a closed digital contour denoted by the ordered set $C = \{c_0, c_1, c_2, \ldots c_{N_C}\}$, where $c_0 = c_{N_C}$. Given C and an error threshold δ, we are required to obtain a polygon P with minimal number of vertices such that $P \subseteq C$ and the maximal distance between P and C is less than or equal to δ. We denote such a polygon by the ordered set $P = \{p_0, p_1, p_2, \ldots p_{N_P}\}$. At this stage, it is assumed that $p_0 = c_0$.

Let $\overrightarrow{p_{k-1}, p_k}$ be the polygon edge that approximates the partial contour $\{c_i = p_{k-1}, c_{i+1}, \ldots c_{i+L} = p_k\}$ containing $(L+1)$ points as shown in Fig. 1. The edge distortion of $\overrightarrow{p_{k-1}\,p_k}$, denoted by $d(p_{k-1} - p_k)$, is defined as the maximum distance between $\overrightarrow{p_{k-1}\,p_k}$ from the partial contour which it approximates. Mathematically,

$$d(p_{k-1}, p_k) = \max_{c_j \in \{c_i = p_{k-1}, c_{i+1}, \ldots c_{i+1} = p_k\}} d'\left(p_{k-1}, p_k, c_j\right), \tag{1}$$

where $d'(p_{k-1}, p_k, c_j)$ denotes the distance of the contour point c_j from the edge $\overrightarrow{p_{k-1}p_k}$.

Let $D(P)$ denote the maximal distance of the polygon P from the contour C. We can express $D(P)$ as the function of polygon edge distortions, as follows.

$$D(P) = \max_{k \in \{1, \ldots N_p\}} d\left(p_{k-1}, p_k\right). \tag{2}$$

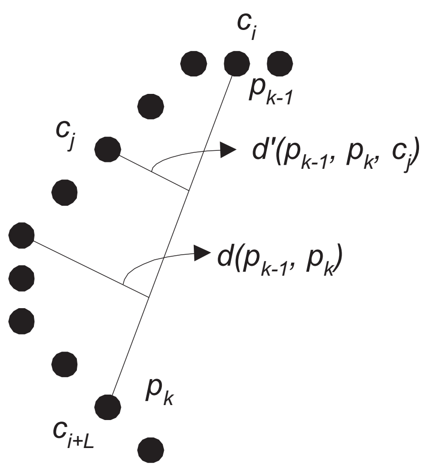

Fig. 1. Computation of edge distortion.

The optimization problem can be stated as,

$$min\ N_p$$

subject to

$$D(P) \leq \delta \tag{3}$$

3 Formulation of the problem in the form of directed acyclic graph

Let a weighted directed acyclic graph with the set of graph vertices V and the set of graph edges E be denoted as $G = (V, E)$. A directed graph edge is denoted by the ordered pair $(v_i, v_j) \in E$, which implies that the edge starts at the vertex v_i and ends at vertex v_j. Let the graph edge set E consist of every possible combination of (v_i, v_j) such that $i < j$. The optimal polygonal approximation problem can be formulated using a DAG such that the vertices and edges of the DAG correspond to possible vertices and edges of the polygonal approximation, respectively. Consider a DAG with $V = C$ such that a directed graph edge (v_i, v_j) represents the polygon edge $\overrightarrow{c_i c_j}$. Furthermore, let the weight $w(v_i, v_j)$ of a graph edge (v_i, v_j) depend on the edge distortion $d(c_i, c_j)$ of the polygon edge $\overrightarrow{c_i c_j}$ as follows.

$$w(v_i, v_j) = \begin{cases} \infty, & \text{if } d(c_i, c_j) > \delta; \\ 1, & \text{if } d(c_i, c_j) \leq \delta. \end{cases} \tag{4}$$

An edge is called an admissible edge if its weight is equal to one; otherwise it is called an inadmissible edge. The length of a path in this DAG becomes infinity if that path includes an inadmissible edge (i.e., an edge corresponding to the polygon edge distortion greater than δ). Therefore the DAG shortest path algorithm will not select these paths. As a result, every path that starts at c_0 and ends at c_{N_c} and has finite length represents a valid polygonal approximation. Therefore, the shortest of all these paths corresponds to the polygon approximation with smallest number of vertices, which is the solution to the problem in (3).

4 Reference Algorithms

The conventional algorithm (CA) for the determination of the optimal polygon approximation is through exhaustive search for the single source shortest path within the DAG [5][7]. Let R_i represent the minimum number of vertices that connect the initial vertex v_0 to ith vertex v_i in the DAG. The conventional optimal algorithm [7] is given as follows.

```
R_0 = 0;
for (i = 1, ... N_C) {
    R_i = ∞
}
for (i = 0, ... N_C − 1) {
    for (j = i + 1, ... N_C) {
        calculate edge distortion d(c_i, c_j);
        if (d(c_i, c_j) > δ) continue j;
        if (R_i + 1 < R_j)
          {R_j = R_i + 1; β_j = i;}
    }
}
```

After execution of the above algorithm, a (R_j, β_j) pair would have been stored at each vertex position. The optimal polygon $P = \{p_0 = c_0, p_1, ...p_{N_P} = c_{N_C}\}$ is then obtained by tracing back the pointers starting from β_{N_C} as follows.

```
N_P = R_{N_C};
p_{N_P} = c_{N_C};
k = N_C;
for (i = N_P, ... 0) {
    p_{i−1} = c_{β_k}
    k = β_k
}
```

The edge distortion $d(c_i, c_j)$ of the edge connecting $c_i \equiv (x_i, y_i)$ to $c_j \equiv (x_j, y_j)$ is computed using (1) as follows. Let $c_k \equiv (x_k, y_k)$ be a boundary point between c_i and c_j. The distance of c_k from the edge $\overrightarrow{c_i c_j}$ is given by [6],

$d'(p_{k-1}, p_k, c_j)$

$$= \frac{|(x_k - x_i)(y_j - y_i) - (y_k - y_i)(x_j - x_i)|}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}. \tag{5}$$

The edge distortion $d(c_i, c_j)$ is then computed as the maximum distance of the boundary points that lie between c_i and c_j, from the edge $\overrightarrow{c_i c_j}$. If there are L boundary points between the two end points of an edge, then L distances need to be computed using (5) while determining the edge distortion of that edge. The conventional algorithm involves the determination of $\frac{N_C(N_C-1)}{2}$ number of edge distortions. Therefore, the conventional algorithm is computationally intensive.

In [6], a sliding window is employed to reduce the numbers of edges in the DAG and thereby reduce the total number of edge distortions that need to be computed. The main idea is to restrict the number of admissible edges for each vertex within a window of fixed length. The length of the window is predefined with an ad-hoc value; the smaller the size of the window, the higher the speed-up. However, smaller window size may not include all the admissible edges and therefore, it is less likely to yield optimal number of vertices. The sliding window algorithm (SWA) provides improvement in speed at the cost of being sub-optimal.

Another fast algorithm called Modified Schuster & Katsaggelos algorithm (MSK) is presented in [7]. The main idea in this algorithm is to declare all the edges that lie beyond the graph vertex at which the polygon edge distortion exceeds twice the error threshold as inadmissible edges. This is equivalent to adapting the window size as we scan along the graph based on the edge distortion observed at the current vertex.

5 Proposed Computationally Efficient Optimal Algorithm

The reference fast algorithms described in the previous section focus only on reducing the number of edges in the DAG; thus, they do not reduce the complexity associated with the computation of edge distortion to determine if that edge is a valid edge. In order to achieve higher speed-up performance, we employ a different approach called cone intersection method to the problem of determining whether an edge is a valid edge.

Suppose we wish to determine the set of all admissible edges starting from a vertex c_i. Consider the point c_{i+1}. Let T_{i+1} be the cone of straight lines formed by a disk of radius δ centered at c_{i+1}. The set of the straight lines from c_i that lie within a distance δ from c_{i+1} are within the cone T_{i+1}. Considering the next scan-point on the boundary, c_{i+2}, the set of the admissible straight lines from c_i that lie within a distance δ from both c_{i+1} and c_{i+2} are within the cone, $T_{i+1} \bigcap T_{i+2}$. For a boundary point c_j, the cone of admissible straight lines that lie within a distance of δ from all the boundary points in the current scan is $S_j = (T_{i+1} \bigcap T_{i+2} \ldots \bigcap T_j)$. At each stage, we test if the current boundary point c_j lies within the cone S_j. If the test succeeds, then the edge $\overrightarrow{c_i c_j}$ is an admissible edge; otherwise, it is an inadmissible edge.

Proposed Algorithm:

$R_0 = 0;$
$for\ (i = 1, \ldots N_C)\ \{$
$\quad R_i = \infty$
$\}$
$for\ (i = 0, \ldots N_C - 1)\ \{$

$for\ (j = i+1, \ldots N_C)\ \{$
calculate the cone S_j of admissible straight lines;
if S_j is empty, break j;
if $\overrightarrow{c_i c_j}$ does not belong to S_j continue j;
if $(R_i + 1 < R_j)$
$\{R_j = R_i + 1; \beta_j = i;\}$
$\}$
$\}$

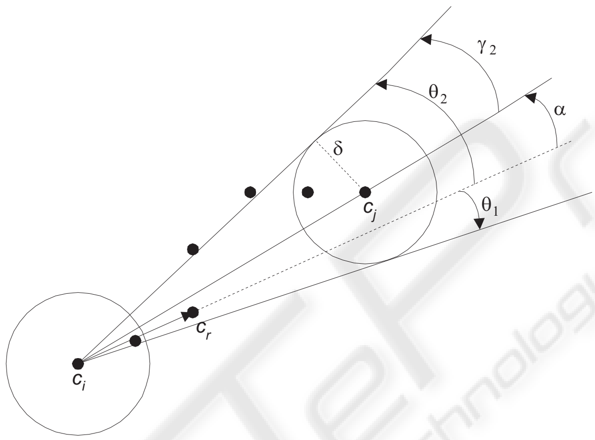

Fig. 2. Illustration of cone intersection method.

We propose the following procedure to calculate the cone S_j of admissible straight lines starting from c_i, and to test if $\overrightarrow{c_i c_j}$ belongs to S_j. The first step is to calculate the angle of cone of straight lines shown in. Fig. 2. The angles are measured from a reference vector $\overrightarrow{c_i c_r}$, where c_r is the first point along the scan that lies at a distance greater than δ from c_i. The edges that lie before the reference vector are all admissible edges.

Let $c_i \equiv (x_i, y_i)$, $c_r \equiv (x_r, y_r)$, and $c_j \equiv (x_j, y_j)$. We define, $({x_j}', {y_j}') = (x_j - x_i, y_j - y_i)$ and $(x'_r, y'_r) = (x_r - x_i, y_r - y_i)$. From Fig. 2, we have,

$$\gamma_2 = arc\ tan \left(\frac{\delta}{\sqrt{{x'_j}^2 + {y'_j}^2 - \delta^2}} \right) \tag{6}$$

$$\alpha = arc\ tan \left(\frac{x'_r y'_j - y'_r x'_j}{x'_r x'_j + y'_r y'_j} \right) \tag{7}$$

(a) (b)

Fig. 3. Original Boundaries: (a) *Kid1* and (b) *Kid2*.

Since *arc tan* is in the principal range between $-\pi/2$ and $\pi/2$, the appropriate value of α under special cases is calculated as follows. We add π to α to get new value of α if any of the following are true: (a) $\alpha < 0$ and $(x'_r y'_j - y'_r x'_j) > 0$ (this condition true if α is in the 2nd quadrant), (b) $\alpha > 0$ and $(x'_r y'_j - y'_r x'_j) < 0$ (this condition true if α is in the 3rd quadrant). The angles θ_1 and θ_2 are calculated as follows.

$$\theta_2 = \alpha + \gamma_2 \tag{8}$$

$$\theta_1 = \alpha - \gamma_2 \tag{9}$$

The algorithm for determining the cone S_j of admissible straight lines starting from c_i, and testing if $\overrightarrow{c_i c_j}$ belongs to S_j is given as follows.

$S_i = \infty;$
if $(j < r)$ {
    $S_j = \infty;$
    $\overrightarrow{c_i c_j} \in S_j$
}
else {
    $T = [\theta_1, \theta_2];$
    $S_j = T \bigcap S_{j-1}.$

    if $(\alpha \in S_j)$
        $(\overrightarrow{c_i c_j} \in S_j);$
    else $(\overrightarrow{c_i c_j} \notin S_j);$

}

6 Experimental Results

The proposed and the reference algorithms are evaluated using the shape boundaries *Kid1* and *Kid2* consisting of 486 and 609 points, respectively. The shape boundaries are shown in Fig. 3. The tolerable approximation error threshold δ is varied in steps from

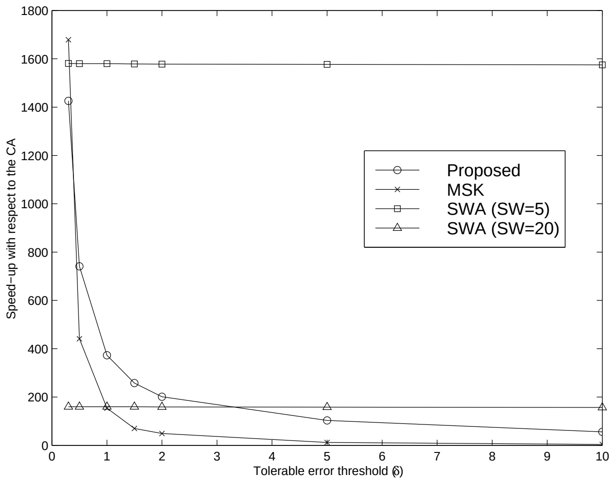

Fig. 4. Comparison of speed-up of proposed optimal algorithm with other algorithms for *Kid1*.

0 to 10. Two separate tests are performed for MSK algorithm by setting window size to 5 and 20. The execution times of each algorithm is obtained from the profiling information generated using the Rational's Quantify (now part of IBM's Purify) profiling tool. The simulations are carried out on a desktop PC with 866 MHz Intel Pentium III processor.

The execution time for CA changes by only negligibly small value when the value of δ is varied. In our experiments, the execution time for CA is approximately 2751 milliseconds for *Kid1* and 6086 milliseconds for *Kid2*. We compare the performance of MSK, SWA and proposed algorithms for fast polygonal approximation with respect to that of CA. The performance is measured in terms of number of polygonal vertices and speed-up factor. Speed-up factor of a fast polygonal approximation algorithm is defined as the ratio of the execution time of the conventional optimal algorithm (CA) to the that of the fast algorithm.

Fig. 4 and Fig. 5 show the comparison of speed-up performance of the proposed algorithm with that of other fast algorithms for *Kid1* and *Kid2*, respectively. Table 1 shows the number of vertices obtained with each algorithm. As compared to CA, the proposed algorithm is more than 350 times faster at $\delta = 1$ and more than 200 times faster at $\delta = 2$. In our experiments, the proposed algorithm is more than 2 times faster than the MSK for $\delta > 0.3$. For δ less than or equal to 0.3, the MSK is about 1.2 times faster than the proposed algorithm; this is because the number of edge distortion computations in the MSK is small when the value of δ is very small. For SW=5, the SWA is always faster than the proposed algorithm; but the results of SWA are always sub-optimal (i.e., the number of vertices are more than those obtained with the optimal algorithms) as shown in Table 1. For SW=20, the SWA is slower than the proposed algorithm for lower values of δ and is faster for higher values of δ; again the results of SWA are sub-optimal for $\delta \geq 1.0$ as shown in Table 1. Due to the fixed window size,

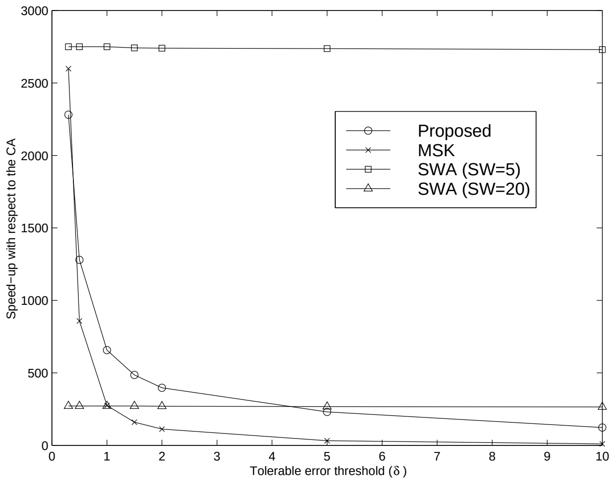

Fig. 5. Comparison of speed-up of proposed optimal algorithm with other algorithms for *Kid2*.

the speed-up of SWA remains nearly the same for all values of δ. Whereas, the speed-up of proposed and MSK algorithms decreases with increasing value of δ.

The polygon approximations of the boundaries obtained with the proposed algorithm are shown in Fig. 6.

7 Conclusion

The proposed algorithm for optimal polygonal approximation is computationally very efficient over a wide range of approximation error. On an average, it is about 450 times faster than the conventional optimal algorithm and about 5 times faster than the MSK algorithm [7]. Due to high speed-up performance, the proposed algorithm is suitable for real-time shape representation and coding applications.

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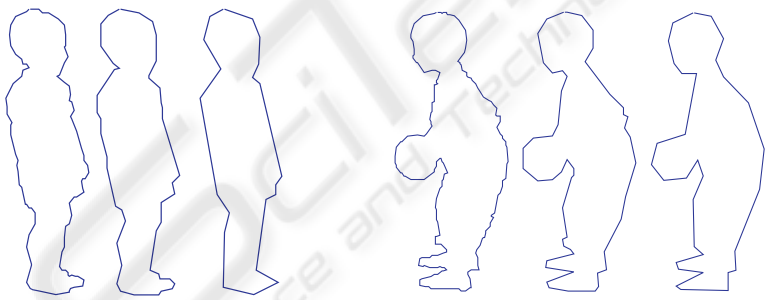
Table 1. Number of vertices in the polygonal approximation.

Tolerable error threshold (δ)	Number of vertices in the polygonal approximation					
	Kid1			*Kid2*		
	CA, MSK, Proposed Algo.	SWA (SW=5)	SWA (SW=20)	CA, MSK, Proposed Algo.	SWA (SW=5)	SWA (SW=20)
0	246	257	246	308	327	308
0.3	211	222	211	254	278	254
0.5	96	133	96	138	182	138
1.0	45	96	47	57	126	59
1.5	30	94	34	42	123	47
2.0	21	94	29	34	122	40
5.0	11	94	24	17	122	31
10.0	6	94	24	10	122	31



(a) (b) (c) (d) (e) (f)

Fig. 6. Polygonal approximations using proposed optimal algorithm. (a)-(c): *Kid1* at $\delta = 0.5$, $\delta = 1$, and $\delta = 2$, respectively. (d)-(f): *Kid2* at $\delta = 0.5$, $\delta = 1$, and $\delta = 2$, respectively.