STABILITY ANALYSIS OF A THREE-TIME SCALE SINGULAR PERTURBATION CONTROL FOR A RADIO-CONTROL HELICOPTER ON A PLATFORM

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Abstract: A stability analysis is conducted on the proposed three-time scale singular perturbation control that is applied to a Radio/Control helicopter on a platform to regulate its vertical position. The control law proposed allows to achieve the desired altitude by either selecting a desired collective pitch angle or a desired angular velocity of the blades.

1 INTRODUCTION

Control of rotatory wing aircrafts represents a very challenging task due to the nonlinearities and inherit instabilities present in such systems. The versatility of rotorcrafts allows them to perform almost any task that no conventional aircraft can do, but this ability is ultimately associated to the degree of stability and control characteristics obtained via automatic control design (Curtis, 2003). These stability and control characteristics come at the expense of the complex control designs that are required to deal with these highly nonlinear aerospace systems. The increased performance requirements of a continuously growing aerospace industry has called for better control designs that can deal with the more complex systems, making linear control techniques insufficient to cope with the industry demands.

During the last decades, a wide range of different nonlinear control techniques have been studied to deal with the nonlinear dynamics of aerospace systems. Some of these techniques include singular perturbation (Kokotović et al., 1986), feedback linearization (Meyer et al., 1984), dynamic inversion (Bugajski et al., 1990; Reiner et al., 1995; Snell et al., 1992), sliding mode control(Sira-Ramirez et al., 1994), or backstepping control methods (Khalil, 1996; Lee and Kim, 2001) to name few. Neural Networks (NN) are also included within the realm of nonlinear control techniques, and extensive work has also been conducted including Adaptive Critic Neural Network (ACNN) based controls (Balakrishnan and Huang, 2001), feedback linearization along with neural-networks as an alternative to gain scheduling (Leiter et al., 1995; Calise et al., 1999), Neural Generalized Predictive Control (NGPC) algorithms capable of real-time control law reconfiguration (Haley and Soloway, 2001), or generic neural flight control and autopilot systems (Bull et al., 2000) to name few.

A basic problem in control design is the mathematical modelling complexity and precision required for the control designs to have a good performance. The modelling of many systems calls for high-order dynamic equations, which for the case of rotorcraft systems represents a unique challenge: in addition to the modelling complexity of aerodynamic surfaces for a wide range of conditions, it is necessary to take into account the added complexity of the rotatory machinery associated to the aerodynamic surfaces. Generally, the presence of parasitic parameters such as small time constants is often the source of a increased order and stiffness of these systems (Naidu and Calise, 2001). The stiffness, attributed to the simultaneous occurrence of slow and fast phenomena, gives rise to time-scales, and the suppression of the small parasitic variables results in degenerated, reduced order systems, called singularly perturbed systems, that can be stabilized separately, and thus simplifying the burden of control design of high-order systems. The literature gives a extended survey of the use of singular perturbed and time-scales control methods in aerospace systems (Naidu and Calise, 2001; Naidu, 2002).

The motivation to this article comes from the work

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of (Sira-Ramirez et al., 1994) that used a dynamical multivariable discontinuous feedback control strategy of the sliding mode type for the stabilization of a nonlinear helicopter model in vertical flight which include the dynamics of the collective pitch actuators. This article analyzes extensively, for the range of desired final values, the stability of the closed loop system for the singular perturbation control law proposed by the authors (Esteban et al., 2005), which was shown to outperform the results presented by (Sira-Ramirez et al., 1994). This article is structured as follows: Section 2 presents the helicopter model used throughout this article, including an analysis of the equilibrium points of the model; Section 3 introduces the singular perturbation control law formulation derived in (Esteban et al., 2005); the stability analysis of the closed loop system is developed in section 4; simulation results of the closed loop system are depicted in Section 5, conclusions and future work are described in Section 6 and figures of the computer simulations are shown in Section 6.

2 MODEL DEFINITION

The helicopter model that will used throughout the remainder of this article is obtained from several technical reports that were written at the University of Purdue (Pallet et al., 1991; Pallet and Ahmad, 1991) that describe the vertical motion of a radio/control helicopter model mounted on a stand as seen in Fig. 1. The model includes the nonlinear vertical motion of the helicopter and the nonlinear dynamics of the collective pitch actuators, which increases considerably the complexity of the model but also depicts a more realistic model. The differential set of equations that describes the vertical motion of the X-Cell 50 (Miniature-Aircraft-USA, 1999) model miniature helicopter are

$$\ddot{z} = K_1(1+G_{eff})C_T\omega^2 - g - K_2\dot{z} - K_3\dot{z}^2 - K_4, \quad (1)$$

where C_T is the thrust coefficient of the helicopter model, ω $(radians)$ is the rotational speed of the rotor blades, z $(meters)$ is the height of the helicopter above the ground, g (m/s^2) is the gravitational acceleration, and G_{eff} models the ground effect, but during the remainder of this article it will be considered negligible ($G_{eff}=0$). The thrust coefficient and the dynamics of the angular velocity of the blades are modelled as

$$C_T = \left[-K_{C1} + \sqrt{K_{C1}^2 + K_{C2}\theta_c}\right]^2 \quad (2)$$

$$\dot{\omega} = -K_5\omega - K_6\omega^2 - K_7\omega^2\sin\theta_c + K_8u_{th} + K_9, \quad (3)$$

where θ_c (rad) is the collective pitch angle of the rotor blades. The dynamics of the collective pitch angle are defined as

$$\begin{aligned}\ddot{\theta}_c &= K_{10}(-0.00031746u_{\theta_c} + 0.5436 - \theta_c) - \\ &\quad K_{11}\dot{\theta}_c - K_{12}\omega^2\sin\theta_c, \end{aligned} \quad (4)$$

where the inputs to the system are the throttle(u_{th}) and the input to the collective servomechanism (u_{θ_c}). The nominal values of the parameters are K_1=0.25, K_2=0.1, K_3=0.1, K_4=7.86, K_5=0.7, K_6=0.0028, K_7=0.005, K_8=-0.1088, K_9=-13.92, K_{10}=800, K_{11}=65, K_{12}=0.1, K_{C_1}=0.03259, K_{C_1}=0.061456, and g=9.81. Equations (1), (3) and (4) can be written into the non linear equations of motion by defining the state space vector as

$$x = \begin{bmatrix} z \\ \dot{z} \\ \omega \\ \theta_c \\ \dot{\theta}_c \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \quad (5)$$

being the resulting nonlinear equations of motion,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2(a_1 + a_2x_4 - \sqrt{a_3 + a_4x_4}) + a_5x_2 + a_6x_2^2 + a_7 \\ \dot{x}_3 &= a_8x_3 + a_{10}x_3^2\sin x_4 + a_9x_3^2 + a_{11} + u_1 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= a_{13}x_4 + a_{14}x_3^2\sin x_4 + a_{15}x_5 + a_{12} + u_2, \end{aligned} \quad (6)$$

where the constants are a_1=5.31×10^{-4}, a_2=1.5364×10^{-2}, a_3=2.82 × 10^{-7}, a_4=1.632 × 10^{-5}, a_5=-K_2, a_6=-K_2, a_7=-g-K_4, a_8=-K_5, a_9=-K_6, a_{10}=-K_6, a_{11}=K_9, a_{12}=0.5436K_{10}, a_{13}=-K_{10}, a_{14}=-K_{12}, and a_{15}=-K_{11}.

2.1 Equilibrium Points Analysis of the Helicopter Model

In order to better understand the behavior of the system, an analysis of the equilibrium points is conducted. The equilibrium points are obtained by setting all the derivatives of system (6) to zero thus yielding the equilibrium equations

$$\begin{aligned} 0 &= x_2 \\ 0 &= x_3^2(a_1 + a_2x_4 - \sqrt{a_3 + a_4x_4}) + a_5x_2 + \\ &\quad a_6x_2^2 + a_7 \\ 0 &= a_8x_3 + a_{10}x_3^2\sin x_4 + a_9x_3^2 + a_{11} + u_1 \\ 0 &= x_5 \\ 0 &= a_{13}x_4 + a_{14}x_3^2\sin x_4 + a_{15}x_5 + a_{12} + u_2. \end{aligned} \quad (7)$$

As seen in the previous section, the system is formed by five state variables, and two control signals, therefore two degrees of freedom are expected, but when conducted the equilibrium points analysis, only one degree of freedom is observed, three equations with four unknowns ($\bar{x}_3, \bar{x}_4, \bar{u}_1, \bar{u}_2$), where the bar denotes the value at equilibrium,

$$0 = \bar{x}_3^2(a_1 + a_2\bar{x}_4 - \sqrt{a_3 + a_4\bar{x}_4}) + a_7 \quad (8)$$

$$0 = a_8\bar{x}_3 + a_{10}\bar{x}_3^2\sin\bar{x}_4 + a_9\bar{x}_3^2 + a_{11} + \bar{u}_1 \quad (9)$$

$$0 = a_{13}\bar{x}_4 + a_{14}\bar{x}_3^2\sin\bar{x}_4 + a_{12} + \bar{u}_2, \quad (10)$$

and the vertical velocity of the helicopter and the collective pitch rate of the blades is equal to zero

($\bar{x}_2 = \bar{x}_5 = 0$). This is caused because the helicopter equilibrium altitude ($\bar{x}_1$) does not show in any of the equilibrium equations, therefore every equilibrium point can be attained at any altitude. This implies that there exists an infinitely number of equilibrium points, and one of the variables needs to be fixed in order to determine a single equilibrium point. The first equilibrium equation, Eq. (8), defines the equilibrium space by selecting a desired value for either $\bar{x}_3$ or $\bar{x}_4$, such that an expression can be determined as a function of the selected desired variable, defined from now on as x_{3_D} or x_{4_D} respectively. The last Eqs. (9-10), define the control signals required for achieving the selected equilibrium points. If the collective pitch angle (x_{4_D}) is selected as the fixed variable, the expressions to determine the values of the other three unknowns as a function of de fixed variable ($\bar{x}_3(x_{4_D})$, $\bar{u}_1(x_{4_D})$ and $\bar{u}_2(x_{4_D})$) can be expressed as,

$$\bar{x}_3 = \pm\sqrt{-\frac{a_7}{a_1 + a_2 x_{4_D} - \sqrt{a_3 + x_{4_D} a_4}}} \quad (11)$$

$$\bar{u}_1 = -a_8\sqrt{-\frac{a_7}{a_1 + a_2 x_{4_D} - \sqrt{a_3 + a_4 x_{4_D}}}} + \frac{a_7(a_{10}\sin x_{4_D} + a_9)}{a_1 + a_2 x_{4_D} - \sqrt{a_3 + a_4 x_{4_D}}} - a_{11} \quad (12)$$

$$\bar{u}_2 = -a_{12} + \frac{a_7 a_{14}\sin x_{4_D}}{a_1 + a_2 x_{4_D} - \sqrt{a_3 + a_4 x_{4_D}}} - a_{13} x_{4_D}. \quad (13)$$

If the angular velocity of the blades (x_{3_D}) is selected as the fixed variable, the expressions to determine the values of the other three unknowns as a function of de fixed variable ($\bar{x}_4(x_{3_D})$, $\bar{u}_1(x_{3_D})$ and $\bar{u}_2(x_{3_D})$) can be expressed as

$$\bar{x}_4 = \frac{a_4 x_{3_D} \pm \sqrt{K_b x_{3_D}^2 + K_c}}{2a_2^2 x_{3_D}} + K_d + \frac{K_e}{x_{3_D}^2}, \quad (14)$$

$$\bar{u}_1 = -a_8 x_{3_D} - x_{3_D}^2(a_{10}\sin\bar{x}_4 + a_9) - a_{11} \quad (15)$$

$$\bar{u}_2 = -a_{13}\bar{x}_4 - a_{14} x_{3_D}^2 \sin\bar{x}_4 - a_{12}, \quad (16)$$

being the coefficients defined by

$$K_b = a_4^2 - 4a_2 a_1 a_4 + 4a_2^2 a_3$$
$$K_c = -4a_2 a_7 a_4$$
$$K_d = -\frac{a_1}{a_2}$$
$$K_e = -\frac{a_7}{a_2}.$$

It can be observed that Eq. (11) has two solutions for the equilibrium rotational speed of the blades ($\bar{x}_3$), but constrained by the physical rotation of the blades, only the positive solution is considered. It is also observed that Eq. (14) has two solutions for the equilibrium collective pitch angle of the blades ($\bar{x}_4$), but it can be checked by substituting both solutions in the original equations (7) that the solution corresponding to the minus sign in front of the square root is a false solution introduced in the previous computations, therefore only the positive solution is considered. Note that in both Eq. (15) and Eq. (15) depend on $\bar{x}_4$ defined in Eq. (14).

3 SINGULAR PERTURBATION FORMULATION

The general two-time scale singular perturbation model formulation is described (Kokotović et al., 1986) as,

$$\dot{x} = f(x, z, \varepsilon, t),\ x(t_0) = x^0,\ x \in R^n \quad (17)$$
$$\varepsilon\dot{z} = g(x, z, \varepsilon, t),\ z(t_0) = z^0,\ z \in R^m, \quad (18)$$

and its quasi-steady-state condition is obtained when $\varepsilon = 0$ thus reducing the dimension of the state space defined in Eqs. (17) and (18) from $n + m$ to n. This quasy-steady state condition of the differential equation that represents the ε-fast dynamics degenerates into the algebraic equation

$$0 = g(\bar{x}, \bar{z}, 0, t), \quad (19)$$

where the bar denotes that the variables belong to a system with $\varepsilon = 0$. The new model is considered in standard form if and only if in a domain of interest, Eq. (19), has $k \geq 1$ distinct real roots (Kokotović et al., 1986):

$$\bar{z} = \bar{\phi}_i(\bar{x}, t),\ i = 1, 2, ..., k. \quad (20)$$

This assumption assures that a well defined n-dimensional reduced model will correspond to each root of Eq. (20). To obtain the i^{th} reduced model, Eq. (20) is substituted into Eq. (17) yielding

$$\dot{\bar{x}} = f(\bar{x}, \bar{z}, \bar{\phi}_i(\bar{x}, t), 0, t),\ \bar{x}(t_0) = x^0, \quad (21)$$

and keep the same initial conditions for the state variable $\bar{x}(t)$ as for $x(t)$. Using singular perturbation techniques causes the dynamics to behave as a multi-time-scale system simplifying considerably the complexity of the dynamics. The slow response is approximated by the reduced model (21), while the discrepancy between the response of the reduced model, (21), and that of the full model (17) and (18), is the fast transient.

3.1 Multi-time Scale Singular Perturbation Model Formulation

(Esteban et al., 2005) showed that a three-time scale helicopter model was intuitively more precise than a two-time scale due to the treatment of the collective pitch angle as a state variable, generally being treated as a control input. The general formulation of the three-time scale singular perturbed systems requieres the system to posses three different time-scales that are defined as

$$\begin{aligned} \zeta\dot{x} &= f(x,y,z,\varepsilon,t) \\ \dot{y} &= g(x,y,z,\varepsilon,t) \\ \dot{z} &= \varepsilon h(x,y,z,\varepsilon,t), \end{aligned} \quad (22)$$

being $0 < \zeta << 1$ and $0 < \varepsilon << 1$. For the helicopter model, the fast dynamics are defined as

$$\begin{aligned} &\zeta\dot{x}_4 = \zeta x_5 \\ &\zeta\dot{x}_5 = -x_4 + c_1 x_3^2 \sin x_4 + c_2 x_5 + c_3 + \\ &c_4 u_2 \end{aligned} \quad (23)$$

while the intermediate-dynamics are

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = x_3^2(a_1 + a_2 x_4 - \sqrt{a_3 + a_4 x_4}) + a_5 x_2 + \\ &a_6 x_2^2 + a_7, \end{aligned} \quad (24)$$

and the slow-dynamics are

$$\dot{x}_3 = \varepsilon\left(c_5 x_3 - x_3^2 \sin x_4 - x_3^2 + c_6 + c_7 u_1\right). \quad (25)$$

The the parasitic constants are chosen to be $\zeta = -\frac{1}{a_{13}} = 0.00125$ and $\varepsilon = \frac{-1}{a_{10}} = 0.0028$, and the constants of the parameters are

$$c_1 = -\frac{a_{14}}{a_{13}} = \zeta a_{14}, \qquad c_2 = -\frac{a_{15}}{a_{13}} = \zeta a_{15}$$

$$c_3 = -\frac{a_{12}}{a_{13}} = \zeta a_{12}, \qquad c_4 = -\frac{1}{a_{13}} = \zeta$$

$$c_5 = -\frac{a_8}{a_{10}} = \varepsilon a_8, \qquad c_6 = -\frac{a_{11}}{a_{10}} = \varepsilon a_{11}$$

$$c_7 = -\frac{1}{a_{10}} = \varepsilon.$$

The control strategy for the three-time scale singular perturbation formulation consists in treating the three different scales as two distinct singular perturbed problems. The first problem considers the fast and intermediate dynamics, and obtains the associated control law that stabilizes the first subsystem using singular perturbation methodology described in the previous section. For this subsystem, the collective pitch angle dynamics (x_4, x_5) are the ζ-fast dynamics and the vertical motion of the helicopter (x_1, x_2) are the ζ-slow dynamics. The second problem considers the intermediate and slow dynamics, being the vertical motion of the helicopter the ε-fast dynamics and the angular velocity of the blades (x_3) the ε-slow dynamics of the system.

3.1.1 Control formulation for the ζ-singular perturbation subsystem

Prior to determine the control law for the ζ-singular perturbation subsystem, a change of variables is required in the ζ-fast dynamics, Eq. (23), to allow solving for the root of the manifold $0 = f(\bar{x}, \bar{y}, 0, t)$. A feedback transform is introduced such that

$$\bar{u}_2 = c_1 x_3^2 \sin x_4 + c_4 u_2,$$

thus rewriting Eq. (23) into

$$\zeta\dot{x}_4 = \zeta x_5 \quad (26)$$

$$\zeta\dot{x}_5 = -x_4 + c_2 x_5 + c_3 + \bar{u}_2. \quad (27)$$

Setting $\zeta = 0$ yields the root for the fast-dynamics,

$$\bar{x}_5 = 0 \quad (28)$$

$$\bar{x}_4 = \bar{u}_2 + c_3. \quad (29)$$

Substituting for $\bar{x}_4$ and $\bar{x}_5$ into the intermediate dynamics generates the reduced degenerated system,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2(a_1 + a_2(\bar{u}_2 + c_3) - \sqrt{a_3 + a_4(\bar{u}_2 + c_3)}) + \\ &\quad a_5 x_2 + a_6 x_2^2 + a_7, \end{aligned} \quad (30)$$

In order to obtain the control law that stabilizes the ζ-subsystem, a series of algebraic substitutions are conducted. Let

$$w^2 = a_3 + a_4(\bar{u}_2 + c_3). \quad (31)$$

and a expression of $\bar{u}_2$ as a function of w can be obtained such as

$$\bar{u}_2 = \frac{w^2 - a_3 - a_4 c_3}{a_4}, \quad (32)$$

and substituting Eq. (31) and (32) into (33) yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2\left(a_1 + a_2\left(\frac{w^2 - a_3 - a_4 c_3}{a_4} + c_3\right) - w\right) + \\ &\quad a_5 x_2 + a_6 x_2^2 + a_7, \end{aligned} \quad (33)$$

which can be simplified into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2\left(c_8 w^2 - w + Ka\right) + a_5 x_2 + \\ &\quad a_6 x_2^2 + a_7, \end{aligned} \quad (34)$$

being

$$c_8 = \frac{a_2}{a_4}, \; K_a = a_1 + a_2 c_3 - \frac{a_2(a_3 + a_4 c_3)}{a_4}.$$

Let $v = c_8 w^2 - w + K_a$, thus Eq. (34) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2 v + a_5 x_2 + a_6 x_2^2 + a_7. \end{aligned} \quad (35)$$

We choose a stable target system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -b_1(x_1 - x_{1_D}) - b_2 x_2, \end{aligned} \quad (36)$$

where b_1, and b_2 are control design parameters that determine the desired time response, and x_{1_D} represents the desired altitude of the helicopter. The control problem can be solved if a v is chosen such that system (35) behaves like the target system defined in (36). The control signal v is therefore chosen to be:

$$v = \frac{-a_6 x_2^2 - a_7 - dx_2 - b_1(x_1 - x_{1_D})}{x_3^2}, \quad (37)$$

where

$$d = b_2 + a_5. \tag{38}$$

The control law u_2 can the obtained tracing back the algebraic substitutions from the final target system to the initial degenerated system such that

$$u_2 = \frac{\bar{u}_2 - c_1 x_3^2 \sin x_4}{c_4}, \tag{39}$$

where $\bar{u}_2$ is

$$\bar{u}_2 = \frac{w^2 - a_3 - a_4 c_4}{a_4}, \tag{40}$$

where w can be obtained solving the quadratic polynomial

$$\begin{aligned} c_8 w^2 - w + K_a &= v \\ c_8 w^2 - w + K_a - v &= 0, \end{aligned} \tag{41}$$

where v is defined by Eq. (37). Solving for the roots of the polynomial in Eq. (41) yields

$$w = \frac{1 \pm \sqrt{1 - 4c_8(K_a - v)}}{2c_8}. \tag{42}$$

It can be checked by substituting in the original equations (7) that the solution corresponding to the minus sign in front of the square root is a false solution introduced in the previous computations. In the following, only the positive root will be considered. The control law for the $u_2(x_1, x_{1_D}, x_2, x_3, x_4)$ is therefore defined by

$$\begin{aligned} u_2 &= K_f \left(1 + \sqrt{1 - 4c_8 (K_a - v)}\right)^2 + \\ & K_g + K_h x_3^2 \sin x_4, \end{aligned} \tag{43}$$

where

$$\begin{aligned} K_a &= a_1 + a_2 c_3 - \frac{a_2(a_3 + a_4 c_3)}{a_4} \\ K_f &= \frac{1}{4 a_4 c_4 c_8^2} \\ K_g &= -\frac{a_3 + a_4 c_3}{a_4 c_4} \\ K_h &= -\frac{c_1}{c_4}. \end{aligned}$$

Results will be discussed in Section 5

3.1.2 Control formulation for the ε singular perturbation subsystem

Once the first control law for the fast-intermediate system is obtained, the second control law to stabilize the intermediate-slow system needs to be determined. The stabilized vertical motion dynamics becomes the ε-fast system, and the angular velocity of the blades is the ε-slow system. Setting the manifold $0 = g(\bar{x}, \bar{y}, 0, t)$. The root of the ε-fast manifold are determined by setting $0 = g(\bar{x}, \bar{y}, \bar{z}, 0, t)$, yielding

$$\begin{aligned} 0 &= x_2 \\ 0 &= x_3^2(a_1 + a_2 x_4 - \sqrt{a_3 + a_4 x_4}) + a_5 x_2 + \\ & a_6 x_2^2 + a_7 \end{aligned}$$

which represents the first two of the equilibrium Eqs. (7). The first equation yields that the vertical velocity of the helicopter is zero for the ε-fast manifold, and the second equation yields an expression that defines the space of configuration for the vertical motion as a function of both x_3 and x_4.

$$x_3^2(a_1 + a_2 x_4 - \sqrt{a_3 + a_4 x_4}) + a_7 = 0, \tag{44}$$

Solving Eq. (44) for both x_3 and x_4, and substituting the associated independent variable by x_{3_D} and x_{4_D} respectively yields

$$\phi_1 = \pm\sqrt{-\frac{a_7}{a_1 + a_2 x_{4_D} - \sqrt{a_3 + x_{4_D} a_4}}} \tag{45}$$

$$\begin{aligned} \phi_2 &= \frac{a_4 x_{3_D} \pm \sqrt{K_b x_{3_D}^2 + K_c}}{2a_2^2 x_{3_D}} + K_d + \\ & \frac{K_e}{x_{3_D}^2}, \end{aligned} \tag{46}$$

where $\phi_1(x_{4_D})$ represents the solution of the rotor blade angular velocity in the ε-fast manifold when a desired collective pitch angle (x_{4_D}) is selected, and $\phi_2(x_{3_D})$ represents the solution of the collective pitch angle in the ε-fast manifold when a desired rotor blade angular velocity (x_{3_D}) is selected. These two expression allow the designer to choose which one of the variables is considered as the second desired state, which is required to define the equilibrium points of the helicopter. Note that both Eqs. (45) and (46) have two distinct solutions depicted by the $\pm$ sign. As it was shown in Section 2.1, it can be observed that Eq. (45) has two solutions for the rotor blade angular velocity in the ε-fast manifold, ϕ_1, but constrained by the physical rotation of the blades, only the positive solution is considered. It is also observed that Eq. (46) has two solutions for the collective pitch angle in the ε-fast manifold, (ϕ_2), but it can be checked by substituting both solutions in the original equations (7) that the solution corresponding to the minus sign in front of the square root is a false solution introduced in the previous computations, therefore in the future only the positive solution will be considered. Once the roots of the manifold $0 = g(\bar{x}, \bar{y}, \bar{z}, 0, t)$ are defined, the control laws can be obtained by substituting Eqs. (45) or (46) into the ε-slow dynamics depending if the control law has to be solved for a desired collective pitch angle, (x_{4_D}), or a desired angular velocity, (x_{3_D}), respectively yielding:

$$\dot{x}_3 = \varepsilon \left(c_5 \phi_1 - \phi_1^2 \sin x_4 - \phi_1^2 + c_6 + c_7 u_1\right), \tag{47}$$

or

$$\dot{x}_3 = \varepsilon \left(c_5 x_3 - x_3^2 \sin \phi_2 - x_3^2 + c_6 + c_7 u_1\right). \tag{48}$$

The control laws are obtained by defining a target system of the form

$$\dot{x}_3 = -\varepsilon b_3 (x_3 - x_{3_D}), \tag{49}$$

where b_3 represents the desired dynamics of the angular velocity of the blades. The control law associated to Eq. (47) for a desired collective pitch angle (x_{4_D}) is

$$u_1(x_{4_D}) = -\frac{c_5\phi_1 - \phi_1^2 \sin x_{4_D} - \phi_1^2 + c_6}{c_7} - b_3(x_3 - \phi_1), \quad (50)$$

and the control law associated to Eq. (49) for a desired angular velocity of the blades (x_{3_D}) is

$$u_1(x_{3_D}) = -\frac{c_5 x_{3_D} - x_{3_D}^2 \sin\phi_2 - x_{3_D}^2 + c_6}{c_7} - b_3(x_3 - x_{3_D}). \quad (51)$$

Prior to analyze the effectiveness of the proposed control laws for different final conditions, it is necessary to define the limits of the set of desired final conditions that will be considered (x_{3_D} and x_{4_D}). For the limits of the angular velocity of the blades, we assumed that the engine can physically generate a maximum angular velocity of $x_{3_{max}} = 180$ rads/sec. For the range of collective pitch angles a maximum collective pitch angle of $x_{4_{max}} = 0.25$ rads. is considered, and the minimum collective pitch angle can be determined analyzing the modelization of the thrust coefficient, Eq. (2), where it can be observed that only collective pitch angles $x_4 > -\frac{K_{C_1}^2}{K_{C_1}} = -\frac{a_3}{a_4}$ will be defined. Analysis of ϕ_1 shows that there is a region within the collective pitch angle defined range, that it is not defined as an attainable desire final condition. This defines two distinctive regions of interest for the collective pitch angle

$$x_{4_{lim_1}} > x_{4_D} > -\frac{a_3}{a_4}$$
$$x_{4_{max}} > x_{4_D} > x_{4_{lim_2}}$$

being $x_{4_{lim_1}}$ and $x_{4_{lim_2}}$ the roots of the denominator of ϕ_1 equal to zero,

$$x_{4_{lim_1}} = \frac{a_4 - 2a_1a_2 - \sqrt{a_4^2 - 4a_4a_1a_2 + 4a_2^2a_3}}{2a_2^2}$$
$$x_{4_{lim_2}} = \frac{a_4 - 2a_1a_2 + \sqrt{a_4^2 - 4a_4a_1a_2 + 4a_2^2a_3}}{2a_2^2},$$

substituting the constants, the ranges are defined as

$$-0.3992 \times 10^{-3} > x_{4_D} > -0.1727 \times 10^{-1}$$
$$0.25 > x_{4_D} > 0.4138 \times 10^{-3}$$

Figure 2 represent the relation of $\phi_1(x_{4_D})$ and $\phi_2(x_{3_D})$ for the ranges of considered desired collective pitch angle and angular velocity of the blades. Analyzing the results of Fig 2 in detail, it can be seen that despite that entire range of desired final conditions above used is defined, it is illogical to consider desired collective pitch angle values $x_{4_D} < 4.8727°$ since it requires angular velocities above 180 rads/sec to define this equilibrium conditions, thus the range of desired collective pitch angle is reduced to $14.3239° > x_{4_D} > 4.8727°$. Results of the proposed control law will be discussed in Section 5

4 STABILITY ANALYSIS OF THE CLOSED LOOP FORMULATION

In this section, the local asymptotic stability of the desired equilibrium points is analyzed for the resultant closed loop system. The indirect method of *Lyapunov* is used: if all the eigenvalues of the Jacobian, evaluated at the equilibrium, are in the open left-hand complex plane, the equilibrium is asymptotically stable. In the following, this condition is checked using the *Routh-Hurwitz* stability criterion (Routh, 1905). A necessary but not sufficient condition for every solution of $D(s) = 0$ in the left-hand complex plane says that all coefficients of the characteristic polynomial must be greater than zero, otherwise the system is unstable. The sufficient condition of the *Routh-Hurwitz* criterion says that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column of the *Routh* table. Therefore all coefficients of the first column must be positive. In order to construct the *Routh* table, the characteristic polynomial of the system to be tested is assumed to be of the form

$$D(s) = a_0 s^k + a_1 s^{k-1} + \cdots + a_{k-1}s + a_k, \quad (52)$$

The *Routh* table corresponding to the $D(s)$ is obtained by constructing the first two rows transcribing the coefficients of $D(s)$ in alternate rows as shown in table 1. Each succeeding row of the table is completed using entries in the two preceding rows, until there are no more terms to be computed. In the left margin are found a column of exactly k numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ for a kth-order system, where *Routh-Hurwitz* coefficients are

$$b_1 = a_2 - \alpha_1 a_3,\ b_2 = a_4 - \alpha_1 a_5$$
$$c_1 = a_3 - \alpha_2 b_2,\ c_2 = a_5 \quad (53)$$
$$d_1 = b_2 - \alpha_3 c_2,\ e_1 = c_2 = a_5,$$

being the α's defined by

$$\alpha_1 = \frac{1}{a_1},\ \alpha_2 = \frac{a_1}{b_1}$$
$$\alpha_3 = \frac{b_1}{c_1},\ \alpha_4 = \frac{c_1}{d_1} \quad (54)$$
$$\alpha_5 = \frac{d_1}{e_1}$$

Table 1: Routh table.

	1 a_1	a_2 a_3	a_4 a_5	$a_6 \cdots$ $a_7 \cdots$
$\alpha_1 = \frac{1}{a_1}$	$b_1 = a_2 - \alpha_1 a_3$	b_2	b_3	$\cdots$
$\alpha_2 = \frac{a_1}{b_1}$	$c_1 = a_3 - \alpha_2 b_2$	c_2	$\cdots$	
$\alpha_3 = \frac{b_1}{c_1}$	$d_1 = b_2 - \alpha_3 c_2$	$\cdots$		
$\alpha_4 = \frac{c_1}{d_1}$	$\cdots$			
$\vdots$				

The *Routh-Hurwitz* criterion states that the roots of $D(s) = 0$ lie in the left half-plane, excluding the imaginary axis, if and only if all the α's are strictly positive. We need now to obtain the coefficients of the characteristic polynomial of the system defined as

$$\begin{aligned} D(s) &= |sI - J| \\ &= a_0 s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + \\ & \quad a_5, \end{aligned} \tag{55}$$

where I is the 5×5 identity matrix, and J represents the Jacobian of the original system, Eq. (6), and defined as

$$\begin{aligned} J(x) &= \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial x_3} & \frac{\partial \dot{x}_1}{\partial x_4} & \frac{\partial \dot{x}_1}{\partial x_5} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial x_3} & \frac{\partial \dot{x}_2}{\partial x_4} & \frac{\partial \dot{x}_2}{\partial x_5} \\ \frac{\partial \dot{x}_3}{\partial x_1} & \frac{\partial \dot{x}_3}{\partial x_2} & \frac{\partial \dot{x}_3}{\partial x_3} & \frac{\partial \dot{x}_3}{\partial x_4} & \frac{\partial \dot{x}_3}{\partial x_5} \\ \frac{\partial \dot{x}_4}{\partial x_1} & \frac{\partial \dot{x}_4}{\partial x_2} & \frac{\partial \dot{x}_4}{\partial x_3} & \frac{\partial \dot{x}_4}{\partial x_4} & \frac{\partial \dot{x}_4}{\partial x_5} \\ \frac{\partial \dot{x}_5}{\partial x_1} & \frac{\partial \dot{x}_5}{\partial x_2} & \frac{\partial \dot{x}_5}{\partial x_3} & \frac{\partial \dot{x}_5}{\partial x_4} & \frac{\partial \dot{x}_5}{\partial x_5} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, \end{aligned} \tag{56}$$

where the coefficients of the Jacobian are as

$$\begin{aligned} b_{22} &= a_5 + 2a_6 x_2 \\ b_{23} &= 2x_3(a_1 + a_2 x_4 - \sqrt{a_3 + a_4 x_4}) \\ b_{24} &= x_3^2 \left(a_2 - \frac{a_4}{2\sqrt{a_3 + a_4 x_4}} \right) \\ b_{33} &= a_8 + 2a_{10} x_3 \sin x_4 + 2a_9 x_3 + \frac{\partial u_1}{\partial x_3} \\ b_{34} &= a_{10} x_3^2 \cos x_4 + \frac{\partial u_1}{\partial x_4} \\ b_{51} &= \frac{\partial u_2}{\partial x_1} \\ b_{52} &= \frac{\partial u_2}{\partial x_2} \\ b_{53} &= 2a_{14} x_3 \sin x_4 + \frac{\partial u_2}{\partial x_3} \\ b_{54} &= a_{13} + a_{14} x_3^2 \cos x_4 + \frac{\partial u_2}{\partial x_4} \\ b_{55} &= a_{15}, \end{aligned} \tag{57}$$

Recalling that there are two possibilities for the control law, as seen by the duality of u_1, Eqs. (50) and (51), there are also two possibilities for $\frac{\partial u_1}{\partial x_3}$ and $\frac{\partial u_1}{\partial x_4}$ as seen bellow,

$$\begin{aligned} \frac{\partial u_1(x_{3_D})}{\partial x_3} &= \frac{\partial u_1(x_{4_D})}{\partial x_3} = -b_3 \\ \frac{\partial u_1(x_{3_D})}{\partial x_4} &= -\frac{a_7 \cos x_4}{(a_1 + a_2 x_{4_D} - \sqrt{a_3 + a_4 x_{4_D}})c_7} \\ \frac{\partial u_1(x_{4_D})}{\partial x_4} &= 0. \end{aligned}$$

The partial derivatives of the control law Eq. (43) are

$$\begin{aligned} \frac{\partial u_2}{\partial x_1} &= 4K_f c_8 \left(1 + \frac{1}{\sqrt{1 - 4c_8(K_a - v)}} \right) \frac{\partial v}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} &= 4K_f c_8 \left(1 + \frac{1}{\sqrt{1 - 4c_8(K_a - v)}} \right) \frac{\partial v}{\partial x_2} \\ \frac{\partial u_2}{\partial x_3} &= 4K_f c_8 \left(1 + \frac{1}{\sqrt{1 - 4c_8(K_a - v)}} \right) \frac{\partial v}{\partial x_3} + \\ & \quad 2K_h x_3 \sin x_4 \\ \frac{\partial u_2}{\partial x_4} &= K_h x_3^2 \cos x_4, \end{aligned}$$

where the partial derivatives of v with respect to the states are defined as

$$\begin{aligned} \frac{\partial v}{\partial x_1} &= -\frac{b_1}{x_3^2} \\ \frac{\partial v}{\partial x_2} &= -\frac{2a_6 x_2 + d}{x_3^2} \\ \frac{\partial v}{\partial x_3} &= \frac{2a_6 x_2^2 + 2a_7 + 2d x_2 + 2b_1(x_1 - x_{1_D})}{x_3^2}. \end{aligned} \tag{58}$$

Substituting Eq. (56) into (55), yields the coefficients of the characteristic polynomial

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -b_{33} - b_{22} - b_{55} \\ a_2 &= -b_{54} + b_{22} b_{55} + b_{22} b_{33} + b_{33} b_{55} \\ a_3 &= -b_{22} b_{33} b_{55} + b_{33} b_{54} + b_{22} b_{54} - b_{52} b_{24} - \\ & \quad b_{53} b_{34} \\ a_4 &= -b_{52} b_{23} b_{34} - b_{24} b_{51} + b_{52} b_{24} b_{33} + \\ & \quad b_{22} b_{53} b_{34} - b_{22} b_{33} b_{54} \\ a_5 &= -b_{51}(b_{23} b_{34} - b_{33} b_{24}). \end{aligned} \tag{59}$$

Once the Jacobian is calculated, in order to proceed with the stability analysis at the desired final conditions, it is necessary to substitute the equilibrium conditions. These conditions, as seen in section 2.1, imply that the vertical velocity of the helicopter and the rate of change of the collective pitch tend to zero ($x_2 = x_5 = 0$), which simplifies substantially the coefficients and the corresponding analysis. Also when the system achieves the desired operating point, $x_1 = x_{1_D}$, $x_2 = x_{2_D}$ and $x_3 = x_{3_D}$. Once the equilibrium points are substituted, the coefficients of the characteristic polynomial only depend on the variation in x_{3_D} and x_{4_D}, and the final desired altitude (x_{1_D}) is irrelevant for the analysis. Figures 3 and 4 represents the variation of the coefficients of the characteristic polynomial and the *Routh-Hurwitz* coefficients respectively of the closed-loop system as the desired collective pitch angle is varied from $14.3239° > x_{4_D} > 4.8727°$. Figures 5 and 6 represents the variation of the coefficients of the characteristic polynomial and the *Routh-Hurwitz* coefficients respectively of the closed-loop system as the desired collective pitch angle is varied from 180 rads/sec> $x_{4_D} > 52.3163$ rads/sec, and it can be seen that for both ϕ_1 and ϕ_2 all the coefficients of the characteristic polynomial are greater than zero, and the the coefficients of the first column are positive thus all the roots of the characteristic polynomial are negative and the closed loop system is stable.

5 SIMULATION RESULTS

The simulations are conducted using a 4_{th} Runge-Kutta fixed step integration method with an integration step of 0.01 seconds. Only a representative of the sensitivity analysis conducted will presented in this article. For further details refer to the results presented in (Esteban et al., 2005). The sensitivity analysis is conducted to variation in desired final values. The initial conditions of the helicopter are kept constant, $x_1(0) = 0.45m$, $x_2(0) = 0.1\ m/sec$, $x_3(0) = 70\ rads/sec$, $x_4(0) = 0.1\ rads$ and $x_5(0) = 0.5\ rads/sec$, while varying the desired final conditions, x_{1_D} and x_{4_D}. Fig. 7 shows the simulation results for desired final altitudes of $0m \leq x_{1_D} \leq 1.25m$, and Fig. 8 shows the simulation results for desired final collective pitch angle of $0.075rads \leq x_{4_D} \leq 0.2rads$. Fig. 7 is divided into four subfigures, where from left to right and top to bottom represent the helicopter altitude, x_1, angular velocity of the blades, x_3, collective pitch angle, x_4, and both control signals, u_1 and u_2. The control laws perform well and the states are driven to the desired final states. A extended range of initial conditions will be studied and presented on the final version of this article.

6 CONCLUSION

The stability analysis conducted on the closed loop system, for the control law, demonstrates the stability of the control law which corroborates the results presented in (Esteban et al., 2005). The stability analysis also demonstrates that both variants of the control law, depending on selecting x_{3_D} or x_{4_D} as one of the desired final values, are stable. The study also demonstrates that the stability and the effectiveness of the control law has no dependence on the final desired altitude (x_{1_D}). Future work might include the study of the actuators saturation and the robustness of the control law to perturbations, both unmodeled dynamics and external disturbances. Future work will also include the extension of this controller to a real system Radio/Control helicopter model on a platform similar to the one presented in this study.

ACKNOWLEDGMENTS

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FIGURES

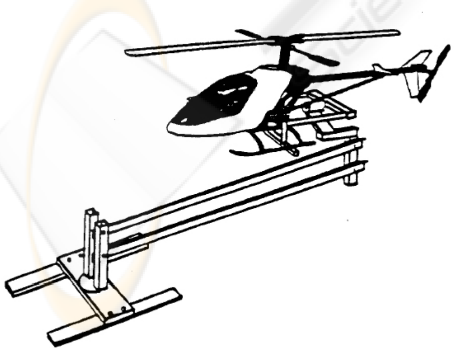


Figure 1: Helicopter mounted on a Stand (Pallet et al., 1991) (Pallet and Ahmad, 1991)

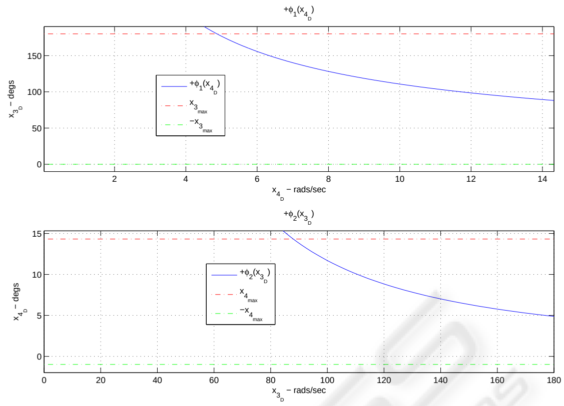


Figure 2: Relation of $\phi_1(x_{4_D})$ and $\phi_2(x_{3_D})$.

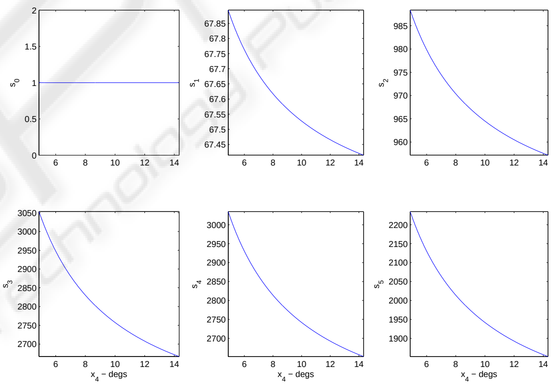


Figure 3: Coefficients for $\phi_1(x_{4_D})$.

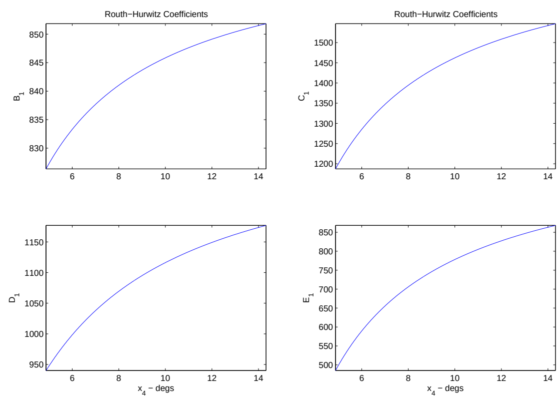


Figure 4: *Routh-Hurwitz* Coefficients for $\phi_1(x_{4_D})$.

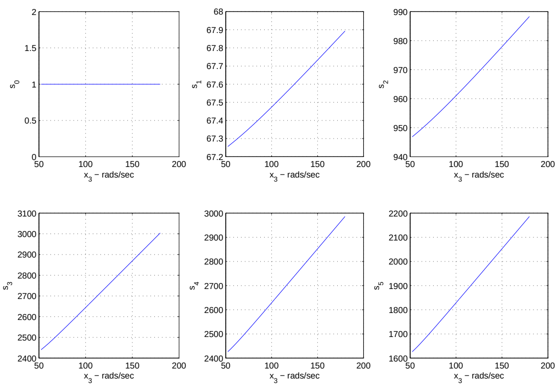


Figure 5: Coefficients for $\phi_2(x_{3_D})$.

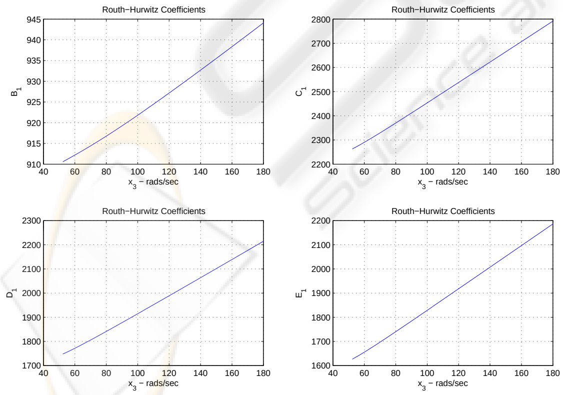
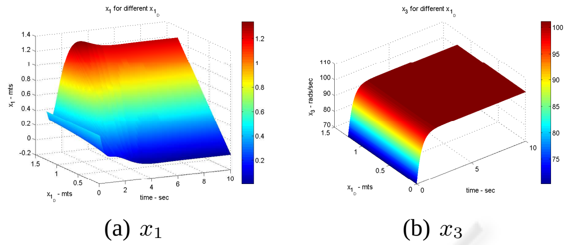


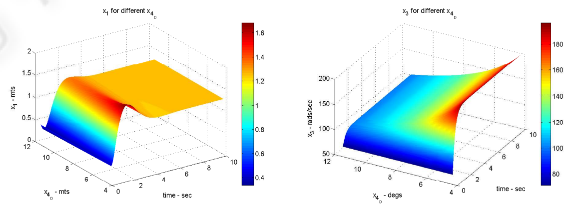
Figure 6: *Routh-Hurwitz* Coefficients for $\phi_2(x_{3_D})$.



(a) x_1 (b) x_3

(c) x_4 (d) $u_1 and u_2$

Figure 7: States and Control Histories For Variable Desired Final Altitude.



(a) x_1 (b) x_3

(c) x_4 (d) $u_1 and u_2$

Figure 8: States and Control Histories For Variable Desired Final Collective Pitch Angle.