ROBUST STABILITY ANALYSIS OF SINGULARLY PERTURBED MAGNETIC SUSPENSION SYSTEMS

§Nan-Chyuan Tsai, Chien-Ting Chen
Department of Mechanical Engineering, National Cheng Kung University, Tainan City, 701,Taiwan

Keywords: Singular perturbation, Two-time-scale systems, Kharitonov Theorem.

Abstract: For a singularly perturbed magnetic suspension system, two kinds of state feedback controllers are synthesized to account for the inherent instability of the open-loop plant with two-time-scale properties. Kharitonov polynomials, extremal vertex and uncertain Nyquist plot are employed to examine the maximum tolerance against system parameters uncertainties such that the stability of the closed-loop system is still retained. Experimental simulations are reported to illustrate the robustness of designed controllers both in stability and performance. At last, Interlacing Theorem is introduced to analyze the stability of uncertain suspension systems *via* the characteristic interval polynomials. It is found that identical results are obtained, in comparison with extremal vertex approach.

1 INTRODUCTION

In general, two-time-scale systems are not uncommon in practice, such as electro-mechanical systems, power engineering and so on. The major feature of this kind of singularly perturbed systems is the eigenvalues of the plant are located distinctly apart into two sectors: one near the origin on complex frequency plane and the other far away. In other words, two-time-scale system can be analyzed as a coupled system composed by a slow subsystem and a fast subsystem.

The singular perturbation technique is employed to decouple the slow subsystem and the fast subsystem by choosing an appropriate coordinate transformation such that the controller for the two-time-scale system can be designed individually by two uncoupled models at first and then composed together. Numerous researchers had applied this concept to synthesize the singularly perturbed industrial control systems. Vournas *et al.* reported to design the generator voltage regulator by singular perturbation method (Vournas, 1995). Flexible robot links have been discussed and presented frequently by this technique (Spong, 1989). Suspension of quarter-car model has been analyzed by two-time-scale model (Salman, 1988). However, the robustness in singularly perturbed suspension has not much been addressed.

In this work, the magnetic suspension system is investigated both in robust stability analysis and in control synthesis. The parasitic parameter, that is crucial in singular perturbation system, is found to be strongly related to the inductance value of the electromagnet. Unfortunately, some of the parameters are not only of small values, but also inherent in uncertainties. Therefore, Kharitonov Theorem is introduced to analyze how robust the closed-loop system will be, from the viewpoint of stability with respect to system parameters variations. Two kinds of controllers are synthesized and compared in experimental simulations. It is concluded that the designed controllers exhibit dramatically robust enough to account for parameter uncertainties up to $\pm 50\%$ variation away from the nominal values, while the performances of the closed-loop system are not detrimentally degraded.

2 PROBLEM FORMULATION

The magnetic suspension system (Fujita, 1995), shown in Figure 1, is inherently unstable so that a closed-loop control strategy is required.

Tsai N. and Chen C. (2005).
ROBUST STABILITY ANALYSIS OF SINGULARLY PERTURBED MAGNETIC SUSPENSION SYSTEMS.
In *Proceedings of the Second International Conference on Informatics in Control, Automation and Robotics - Signal Processing, Systems Modeling and Control*, pages 166-172
DOI: 10.5220/0001188001660172

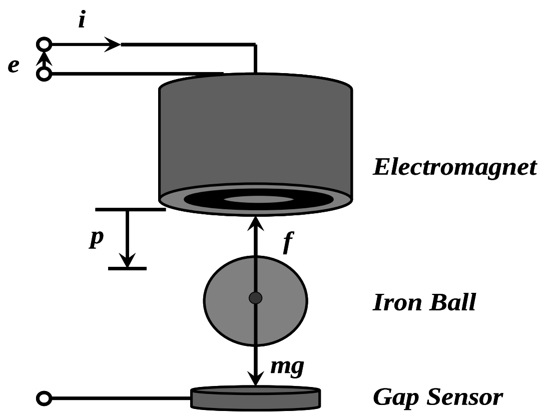

Figure 1: Magnetic suspension system.

2.1 Equation of Motion

The single degree-of-freedom dynamic model of magnetic suspension can be easily described as follows:

$$m\frac{d^2 p}{dt^2} = mg - f \tag{1}$$

$$f = k\left(\frac{i}{p + p_0}\right)^2 \tag{2}$$

$$Ri + L\frac{di}{dt} = e \tag{3}$$

where f is the magnetic control force that is proportional to the square of coil current, $i(t)$, and inversely proportional to the square of total air gap, $p(t)+ p_0$. k is a constant. p_0 is denoted as the offset determined by the measurement instruments and sensor locations. e represents the exerted control voltage applied *via* amplifiers set on the coil of electromagnets. The interested parameters are tentatively considered as constants and listed in Table 1. The linearized state space model can be obtained by taking Taylor's Expansion around ($P+p_0$, I). I is the steady-state control current as the air gap reaches ($P + p_0$) that is the quasi-steady-state value at which the gravity of the control force is balanced.

$$\dot{X} = AX + Bu \tag{4}$$

$$y = CX \tag{5}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2kI^2}{m(P+p_0)^3} & 0 & -\frac{2kI}{m(P+p_0)^2} \\ 0 & 0 & \frac{-R}{L} \end{bmatrix} \tag{6}$$

$$B^T = \begin{bmatrix} 0 & 0 & \frac{1}{L} \end{bmatrix} \tag{7}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{8}$$

$$X^T = \begin{bmatrix} \hat{p} & \dot{\hat{p}} & \hat{i} \end{bmatrix} \tag{9a}$$

$$\hat{p} = p(t) - (P + p_0) \tag{9b}$$

$$\hat{i} = i(t) - I \tag{9c}$$

The measurement is the incremental displacement, $\hat{p}$, only, experimentally available from an eddy-current gap sensor.

Table 1: Parameters

Parameter	Symbol	Value
Mass of the iron ball	m [kg]	1.75
Steady-state gap between the magnet and the iron ball	P [m]	2×10^{-2}
Steady-state current of the electromagnet	I [A]	1.06
Inductance of the electromagnet	L [H]	5.08×10^{-2}
Resistance of the electromagnet	R [Ω]	23.2
Constants determined by experiment	p_0 [m]	10-4
Coefficient of the electromagnetic force	k [Nm^2/A^2]	2.9×10^{-4}

2.2 Singular Perturbation System

From Eq.(4) and the actual experimental data in Table 1, it is obvious to find that the poles, { ± 7.5355, -456.6929}, of the studied open-loop magnetic suspension system are located into two groups that are far away from each other. That is, the pole, "-456.6929", is about 65 times in distance from the origin of complex frequency plane, compared with the poles, {7.5355, -7.5355}. It implies that in time domain the plant has the two-time-scale property with respect to the state variables. Hence, Eq.(4)-(9) can be rewritten as the standard form of singular perturbation model as follows (Kokotovic, 1986):

$$\dot{x} = A_{11}x + A_{12}z + B_1 u \tag{10}$$

$$\varepsilon \dot{z} = A_{21}\, x + A_{22}\, z + B_2 u \tag{11}$$

$$y = C_1\, x + C_2\, z \tag{12}$$

$$A_{11} = \begin{bmatrix} 0 & 1 \\ \dfrac{2kI^2}{m(P+p_0)^3} & 0 \end{bmatrix} \tag{13a}$$

$$A_{12}^T = \begin{bmatrix} 0 & -\dfrac{2kI}{m(P+p_0)^2} \end{bmatrix} \tag{13b}$$

$$A_{21} = [0 \quad 0] \tag{13c}$$

$$A_{22} = [-R] \tag{13d}$$

$$B_1^T = [0 \quad 0] \tag{13e}$$

$$B_2^T = 1 \tag{13f}$$

$$C_1 = [1 \quad 0] \tag{13g}$$

$$C_2 = 0 \tag{13h}$$

$$\varepsilon = L \tag{13i}$$

$$x^T = [\hat{p} \quad \dot{\hat{p}}] \tag{13j}$$

$$z = \hat{i} \tag{13k}$$

where ε is called as the perturbation parameter that is assumed as a positive scalar but, to some extent, close to zero. The reduced model can be constructed by letting $\varepsilon = 0$.

$$\dot{x}_s(t) = A_s x_s(t) + B_s u_s(t) \tag{14a}$$

$$y_s(t) = C_s x_s(t) + D_s u_s(t) \tag{14b}$$

$$z_s = -A_{22}^{-1}(A_{21}\, x_s + B_2\, u_s) \tag{14c}$$

where

$$A_s = A_{11} - A_{12} A_{22}^{-1} A_{21} \tag{15a}$$

$$B_s = B_1 - A_{12} A_{22}^{-1} B_2 \tag{15b}$$

$$C_s = C_1 - C_2 A_{22}^{-1} A_{21} \tag{15c}$$

$$D_s = -C_2 A_{22}^{-1} B_2 \tag{15d}$$

x_s and z_s are the state variables of reduced model. y_s is the output of the slow subsystem. The overall controller can be designed individually on the bases of slow subsystem model and fast subsystem. For example, if the state feedback control strategy is taken, then:

(a) on the base of slow subsystem,

$$u_s = H_0\, x_s \tag{16}$$

That is, to design u_s is based on Eq.(14a) and Eq.(14b) only.

(b) on the base of fast subsystem,

$$u_f = H_2\, z_f \tag{17}$$

That is, to design u_f is based on Eq.(14c) and the fast subsystem defined as follows:

$$\varepsilon \dot{z}_f = A_{22}\, z_f + B_2\, u_f \tag{18a}$$

$$y_f = C_2\, z_f \tag{18b}$$

where $z_f = z - z_s$, $u_f = u - u_s$.

To sum up, the overall state feedback can be constructed by direct composition by addition.

$$\begin{aligned} u &= u_S + u_f \\ &= H_0\, x_s + H_2\, z_f \\ &= H_0\, x \\ &\quad + H_2\,[z + A_{22}^{-1}(A_{21}\, x + B_2\, H_0\, x)] \\ &= H_1\, x + H_2\, z \end{aligned} \tag{19}$$

where $H_1 = (I + H_2 A_{22}{}^{-1} B_2) H_0 + H_2 A_{22}{}^{-1} A_{21})$

Its schematic control loops are shown in Figure 2.

3 CONTROLLER DESIGN ON SINGULARLY PERTURBED SUSPENSTION SYSTEMS

It has been well known that an appropriate controller design on the reduced perturbation model can be applied on the actual suspension systems. The errors, of states or system output, are restricted in first-order zero approximation, i.e., $O(\varepsilon)$, due to effect caused by reduction from exact model as long as the fast subsystem matrix of perturbed state-space model, $A_{22}(x, z, t)$, is Hurwitz (Saksena, 1984). In other words, partial state feedback can be utilized to ensure the asymptotic stability and performance of the actual system. If the state feedback law is described as follows:

$$u = [H_1 \quad H_2][x^T \quad z^T]^T \tag{20}$$

Two kinds of feedback controllers are designed and compared in this work: *Eigenvalues Assignment* and *Near-optimal approach.*

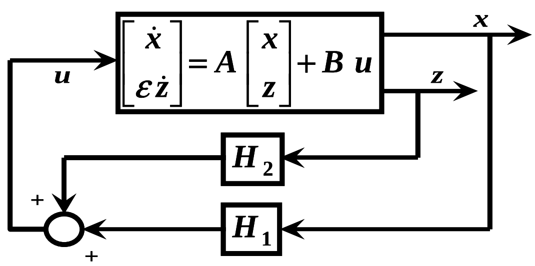

Figure 2: Composite state feedback.

3.1 Eigenvalue Assignment

Under assumption of controllability of reduced slow subsystem, the eigenvalues can be assigned anywhere on the complex frequency plane. At least the unstable poles can be moved to the stable region if the slow subsystem is stabilizable. The feedback gain matrix is denoted as *H* that will be used later.

3.2 Near-Optimal Approach

A near-optimal linear quadratic regulator is designed against the performance index:

$$J_s = \frac{1}{2}\int_0^\infty (y_s^T\, y_s + u_s^T\, R u_s)\, dt \qquad (21)$$

where the subscript"s" represents the approach is undertaken in slow-state model. The associated steady-state matrix Riccati Equation can be obtained by traditional optimization methodology.

$$\begin{aligned} 0 = &-P_s\,(A_s - B_s\, R_0^{-1}\, D_s^T\, C_s) \\ &-(A_s - B_s\, R_0^{-1}\, D_s^T\, C_s)^T\;\; P_s \\ &+P_s\, B_s\, R_0^{-1}\, B_s^T\, P_s \\ &-C_s^T\,(I - D_s\, R_0^{-1}\, D_s^T)C_s \end{aligned} \qquad (22)$$

where

$$R_0 = R + D_s^T\; D_s \qquad (23)$$

Therefore, the control command is generated by the feedback law:

$$u_s = -R_0^{-1}\,(D_s^T\, C_s + B_s^T\; P_s)\, x_s \qquad (24)$$

4 SIMULATION RESULTS

As a gap sensor, a standard induction probe of eddy-current type is placed closely near the bottom of the iron ball in Figure 1. An electromagnet, of "*EI*" shape in geometry, is used to generate magnetic force near the top of the controlled mass. A digital signal processor DSP-based real-time controller is implemented with TMS320C240. The data acquisition board MSP-77230 consists of a set of 12-bit A/D and D/A converters. With non-zero initial conditions, i.e., the state has an initial deviation, the closed-loop suspension system is regulated to be zero within 0.5 second either by near-optimal controller or eigenvalue assignment design, shown in Figure 3a. The associated required control is plotted in Figure 3b. From the viewpoint if stiffness of closed-loop system, a unit step response is examined and shown in Figure 4. quick response.

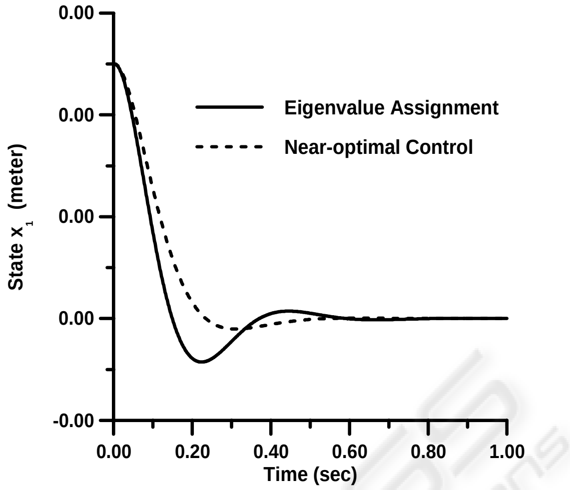

Figure 3a: Time response under non-zero initial condition

Though the performance of the closed-loop system is degraded, subjected to ±50% parameter variation in inductance L, under the identical LQ Controller designed on the base of nominal value, it is not detrimental and exhibits strongly robust, shown in Figure 5.

5 ROBUST STABILITY ANALYSIS

The main reasons that cause singular perturbation are: (i) presence of relatively small parasitic parameters, and (ii) inclusion of hign gain control loops. In this study on magnetic suspension systems, it is evident that the inductance value of the electro-magnet is relatively small so that the singular perturbation problem emerges. Even worse is that the inductance value changes in time. Practically, the

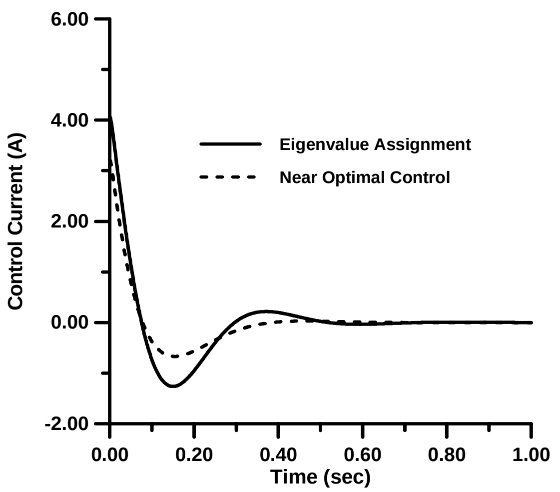

Figure 3b: Control current for linear quadratic regulation

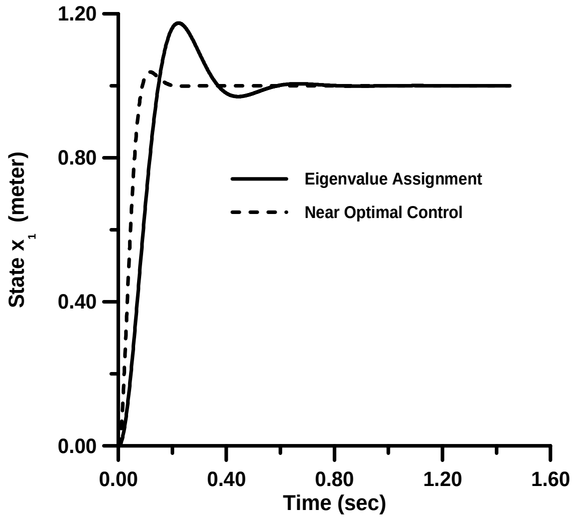

Figure 4: Step response

variation of the inductance can be constrained by a preset pair of upper and lower limits. The parameters uncertainties and resulted robust stability problems are hereby to be investigated as follows.

Applying the concepts of polynomial vertex and Kharitonov segments (Bhattacharyya, 1995), totally four Kharitonov Polynomials are to be simultaneously examined to determine the closed-loop stability region against inductance uncertainty. The *Eigenvalue Assignment* approach is taken as an illustrative example in this report to analyze the stability robustness of the closed-loop system.

The closed-loop system matrix under state feedback can be expressed in the form:

$$\hat{A} = (A - BH) \tag{25}$$

Since the open-loop is a third-order system, the characteristic polynomial of the closed-loop system

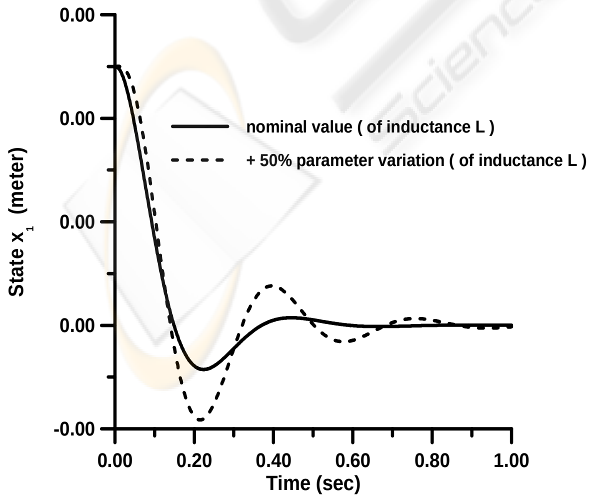

Figure 5: Robust performance of regulation

can be described as follows:

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 \tag{26}$$

If each of the parameters of the characteristic polynomial varies between two limits, i.e., $\delta_0 \in [\alpha_0, \beta_0]$, $\delta_1 \in [\alpha_1, \beta_1]$, $\delta_2 \in [\alpha_2, \beta_2]$, $\delta_3 \in [\alpha_3, \beta_3]$, then the four Kharitonov Polynomial can be obtained:

$$K^1(s) = \alpha_0 + \beta_1 s + \beta_2 s^2 + \beta_3 s^3 \tag{27a}$$

$$K^2(s) = \alpha_0 + \beta_1 s + \beta_2 s^2 + \alpha_3 s^3 \tag{27b}$$

$$K^3(s) = \beta_0 + \alpha_1 s + \alpha_2 s^2 + \beta_3 s^3 \tag{27c}$$

$$K^4(s) = \beta_0 + \beta_1 s + \alpha_2 s^2 + \alpha_3 s^3 \tag{27d}$$

These four extremal polynomials are the complete independent characteristic polynomials to be examined for ensurance of stability of closed-loop uncertain systems. The other twelve extremal polynomials are proved redundant and can be dumped at all (Bhattacharyya, 1995).The effect of parameter uncertainties in Nyquist plot of closed-loop system are shown in Figure 6 and Figure 7, with $\pm$5% and $\pm$50% variation each, with respect to the nominal parameter value, respectively. These two figures conclude that the controller, designed by *Eigenvalue Assignment*, is robust in stability, with maximum tolerance of $\pm$ 50% parameter variations. In other words, when the parameter uncertainty exceeds beyond $\pm$ 50%, the closed-loop control system becomes unstable.

Another approach is to apply *Interlacing Theorem* (Bhattacharyya, 1995). The odd-order and even-order extremal polynomials of the closed-loop systems are defined as follows:

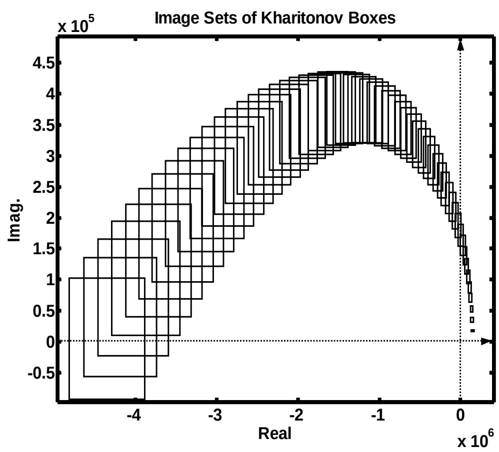

Figure 6: Nyquist plot for uncertainties up to $\pm$5%.

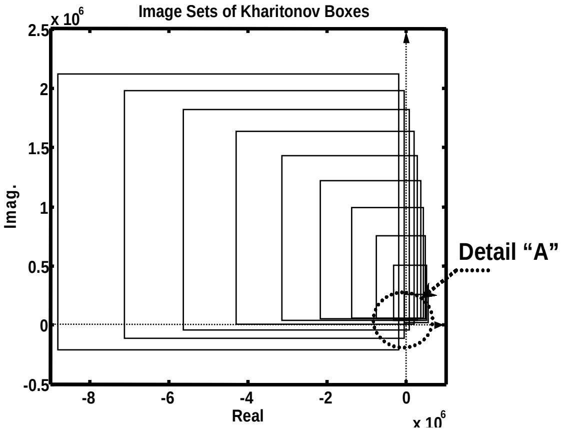

Figure 7a: Nyquist plot for uncertainties up to ± 50%.

$$K_{\max}^{even}(s) = \beta_0 + \alpha_2\, s^2 \quad (28a)$$

$$K_{\min}^{even}(s) = \alpha_0 + \beta_2\, s^2 \quad (28b)$$

$$K_{\max}^{odd}(s) = \beta_1\, s + \alpha_3\, s^3 \quad (28c)$$

$$K_{\min}^{odd}(s) = \alpha_1\, s + \beta_3\, s^3 \quad (28d)$$

Let $s = j\,w$, the Equation set (28) can be rewritten in real or imaginary part as follows:

$$K_{\max}^{even}(w) = K_{\max}^{even}(j\,w) = \beta_0 - \alpha_2\, w^2 \quad (29a)$$

$$K_{\min}^{even}(w) = K_{\min}^{even}(j\,w) = \alpha_0 - \beta_2\, w^2 \quad (29b)$$

$$K_{\max}^{odd}(w) = \frac{K_{\max}^{odd}(j\,w)}{j\,w} = \beta_1 + \alpha_3\, w^2 \quad (29c)$$

$$K_{\min}^{odd}(w) = \frac{K_{\min}^{odd}(j\,w)}{j\,w} = \alpha_1 + \beta_3\, w^2 \quad (29d)$$

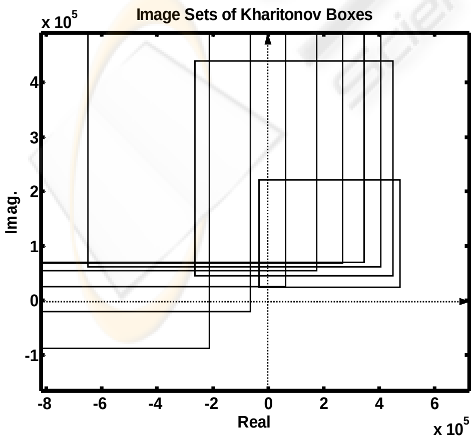

Figure 7b: Detail of "A".

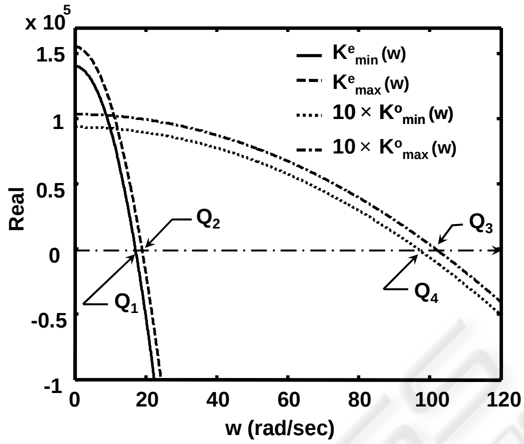

Figure 8: Interlacing of extremal polynomials.

The values of the above four extremal polynomials versus frequency are shown in Figure 8. The intersections of extremal polynomials, $K_{\max}^{even}(w)$, $K_{\min}^{even}(w)$, $K_{\max}^{odd}(w)$ and $K_{\min}^{odd}(w)$, and frequency axis are Q_1, Q_2, Q_3 and Q_4 respectively. According to Interlacing Theorem, the inequality $0<Q_1<Q_2<Q_3<Q_4$ implies that the closed-loop system retains stable under the parameter uncertainties ± 5% each. This result is identical to the one from uncertain Nyquist plot.

6 CONCLUSION

The inductance value in magnetic suspension system plays a crucial role for singular perturbation analysis and synthesis. Since the two-time-scale properties dominate the extent of singularly perturbed stability and performance, the robustness with respect to the perturbation parameter has to be examined. The Kharitonov Polynomials and Interlacing Theorem both verifies that the controller design, either by eigenvalue assignment or near-optimal approach, would retain robust in stability. It has been also proved by experimental simulations. The performance, of the closed-loop system in the worst case of ± 50% inductance variation, is not greatly deteriorated. This implies that the performance robustness is also achieved.

REFERENCES

Bhattacharyya, S. P., Chapellat, H., Keel, L. H. 1995.Robust Control: The Parametric Approach, Prentice Hall PTR.

Chow, J. H., Kokotovic, P. V. 1976. A Decomposition of Near-Optimal Regulators for Systems with Slow and Fast Modes. In IEEE Trans. Automat.Contr. vol. AC-21, pp. 701-705.

Fujita, M., Namerikawa, T., Matsumura, F., Uchida, K. 1995.μ-Synthesis of an Electromgnetic Suspension System. In IEEE Trans. Automat Contr. vol. 40, pp. 530-536.

Khorrami, F., Ozguner, U. 1988. Perturbation method in control of flexible link manipulators. IEEE. pp.310-315

Kokotovic, P. V., Khalil, H. K., O'Reilly, J. O. 1986. Singular Perturbation Methods in Control : Analysis and Design, Academic Press. New York.

Lewis, F. L., Vandegrift, M., 1993. Flexible robot arm control by a feedback linearization/singular perturbation approach. In IEEE International Conference on Robotics and Automation. vol. 3, pp.729-736.

Salman, M. A., Lee, A. Y., Boustany, N. M. 1988. Reduced order design of active suspension control. In Proceedings of the IEEE Conference on Decision and Control Including The Symposium on Adaptive Processes. pp.1038-1043.

Saksena, V. R., O'Reilly, J. O., Kokotovic, P. V. 1984. Singular perturbations and time-scale methods in the control theory. In Automatic. vol. 20, pp.273-293.

Spong, M. W, 1989. On the force control problem for flexible joint manipulators. In IEEE Transactions on Automatic Control. vol. 34, pp.107-111.

Vournas, C. D., Sauer, P. W., and Pai, M. A, 1995. Time-scale decomposition in voltage stability analysis of power systems. In IEEE Conference on Decision and Control, vol. 4, pp.3459-3464.